3 examples of when global sections of line bundles is interesting.

Informal working seminar, 02.06.2010

Melbourne Univ.

Polytopes to Toric varieties

Let $\mathfrak{h}_{\mathbb{Z}}^*$ be a lattice in $\mathfrak{h}_{\mathbb{R}}^* \cong \mathbb{R}^n$.

A set $\Omega \subseteq \mathfrak{h}_{\mathbb{R}}^*$ is convex if Ω satisfies:

if $x, y \in \Omega$ and $\alpha \in [0,1]_{\mathbb{R}}$ then $\alpha x + (1-\alpha) y \in \Omega$.

Let $S \subseteq \mathfrak{h}_{\mathbb{R}}^*$. The convex hull of S is the subset $\mathrm{conv}(S)$ of $\mathfrak{h}_{\mathbb{R}}$ such that

(a) $\mathrm{conv}(S)$ is convex and $\mathrm{conv}(S) \supseteq S$,

(b) If C is convex and $C \supseteq S$ then $C \supseteq \mathrm{conv}(S)$.

An integer polytope P is the convex hull of a finite subset of $\mathfrak{h}_{\mathbb{Z}}^*$.

The normal fan to P is

$$\Delta_P = \{ \sigma_Q^\vee \mid Q \text{ is a face of } P \}$$

where

$$\sigma_Q^\vee = \{ v^\vee \in \mathfrak{h}_{\mathbb{R}} \mid \langle u, v^\vee \rangle \geq \langle u', v^\vee \rangle \text{ for } u \in Q, u' \in P \}.$$

Let

$$\mathbb{C}[\sigma^\vee \cap \mathfrak{h}_{\mathbb{Z}}] = \mathbb{C}\text{-span}\{ X^\lambda \mid \lambda \in \sigma^\vee \cap \mathfrak{h}_{\mathbb{Z}} \}$$

with $X^\lambda X^\mu = X^{\lambda+\mu}$. and let

$$U_{\sigma^\vee} = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap \mathfrak{h}_{\mathbb{Z}}]).$$

The toric variety of Δ is

$$X(\Delta) = \bigcup_{\sigma \in \Delta} U_\sigma \text{ with } U_{\sigma_1^\vee} \text{ and } U_{\sigma_2^\vee} \text{ glued along } U_{\sigma_1^\vee \cap \sigma_2^\vee}$$

Let

$\tau_1^\vee, \dots, \tau_d^\vee$ be the rays of Δ

$\alpha_1^\vee, \dots, \alpha_d^\vee$, ~~the~~ with $\alpha_i^\vee$ the first lattice point along $\tau_i^\vee$.

$D_i = \overline{O_{\tau_i^\vee}}$

and a_i be such that

$$P = \{ u \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle u, \alpha_i^\vee \rangle + a_i \geq 0 \text{ for } 1 \leq i \leq d \}$$

Then

$D = a_1 D_1 + \cdots + a_d D_d$ is a divisor on $X(\Delta)$ that corresponds to a line bundle $\mathcal{L}$ on $X(\Delta)$.

There is a bijection

$$\{\text{integer polytopes}\} \longleftrightarrow \left\{ \begin{array}{l} \text{pairs } (X, \mathcal{L}) \text{ where} \\ X \text{ is a toric variety and} \\ \mathcal{L} \text{ is an ample line bundle on } X \end{array} \right\}$$

Further let

$\hat{X}^{\mathcal{L}}_k$ be a basis of $H^0(X, \mathcal{L}^{\otimes k})$.

Then

$$\mathrm{Card}(\hat{X}_{\mathcal{L}^k}) = \mathrm{Card}(kP \cap \mathfrak{h}^*_{\mathbb{Z}}).$$

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Flag varieties

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(1)

G complex reductive algebraic group

$\cup|$

B Borel subgroup

$\cup|$

T maximal torus.

Theorem ~~Let X^λ be a dominant weight of T~~

The irreducible finite dimensional G-modules are

$$H^0(G/B, \mathcal{L}_\lambda) \quad \text{for dominant integral weights } \lambda.$$

where $\mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda$ is a line bundle on G/B

and $\mathbb{C}_\lambda$ is the one dimensional B-module coming from the character $X^\lambda : T \to \mathbb{C}^\times$ indexed by λ.

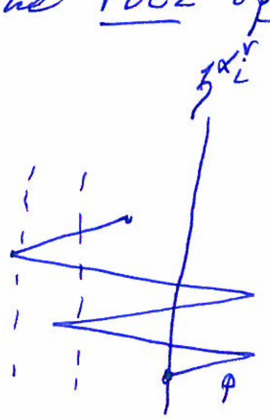
Let $(W_0, \mathfrak{h}^*_\mathbb{Z})$ be the $\mathbb{Z}$-reflection group corresponding to (G,T) and let

C be the chamber of $\mathfrak{h}^*_\mathbb{R}$ corresponding to B

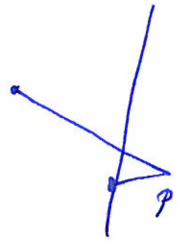
Let $\mathfrak{h}^{\alpha_1^\vee}, \ldots, \mathfrak{h}^{\alpha_n^\vee}$ be the walls of C.

A path in $\mathfrak{h}^*_\mathbb{R}$ is a piecewise linear map $p: [0,1] \to \mathfrak{h}^*_\mathbb{R}$ such that $p(0) = 0$ and $p(1) \in \mathfrak{h}^*_\mathbb{Z}$.

The <u>root operators</u> $\tilde{f}_1, \dots, \tilde{f}_n$ are given by



and



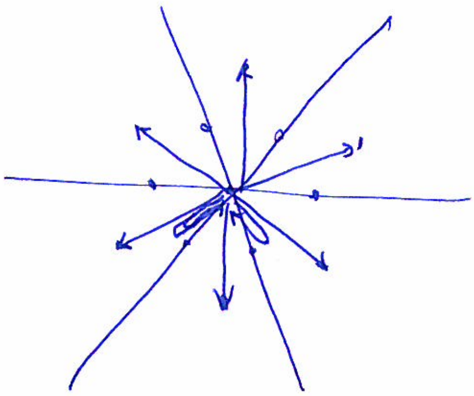
$\tilde{f}_i p = 0$.

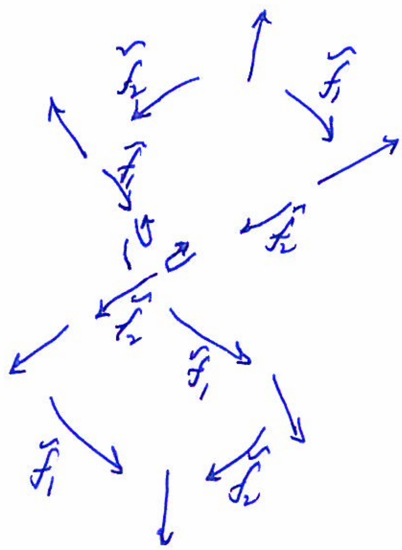
Let $B(\lambda)$ be the crystal generated by p_λ where $p_\lambda(0) = 0$, $p_\lambda(1) = \lambda$ and $p_\lambda \subseteq \overline{C}$.

Then

$$\mathrm{char}\left(H^0(G/B, \mathcal{L}_\lambda)\right) = \sum_{\mu \in \mathfrak{h}_{\mathbb{Z}}^*} \mathrm{Card}(B(\lambda)_\mu)\, X^\mu .$$

<u>Example</u> $G = SL_3(\mathbb{C})$





Let $LG^\vee = G^\vee(\mathbb{C}((t)))$

$$\begin{array}{ccc} \cup\!\!\!\mid & & \\ K^\vee = G^\vee(\mathbb{C}[[t]]) & \xrightarrow{t=0} & G^\vee(\mathbb{C}) \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ I^\vee = \Phi^{-1}(B^\vee) & \longrightarrow & B^\vee \end{array}$$

$LG^\vee/K^\vee$ is the <u>loop Grassmannian</u>

$LG^\vee/I^\vee$ is the <u>affine flag variety</u>

Let $W = W_0 \ltimes \mathfrak{h}_{\mathbb{Z}}^* = \{ w X^{\lambda^\circ} \mid w \in W_0, \lambda^\circ \in \mathfrak{h}_{\mathbb{Z}}^* \}$

with $X^{\lambda^\circ} X^{\mu^\circ} = X^{\lambda^\circ + \mu^\circ}$ and $w X^{\lambda^\circ} = X^{w\lambda^\circ} w$.

Then

$$LG^\vee = \bigsqcup_{\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}^+} K^\vee t_{\lambda^\circ} K^\vee \qquad LG^\vee = \bigsqcup_{\mu^\vee \in \mathfrak{h}_{\mathbb{Z}}} U^- t_{\mu^\circ} K^\vee$$

$$LG^\vee = \bigsqcup_{w \in W_0} I^\vee w I^\vee \qquad LG^\vee = \bigsqcup_{v \in W} U^- v I^\vee.$$

and the <u>MV-intersections</u> are

$$K^\vee t_{\lambda^\circ} K^\vee \cap U^- t_{\mu^\circ} K^\vee$$

and

$$I^\vee w I^\vee \cap U^- v I^\vee$$

An <u>MV-cycle</u> is an irreducible component of

$\overline{K^\vee t_\lambda K^\vee \cap U^- t_\mu K^\vee}$ in $LG^\vee/K^\vee$.

Then, let $\hat{G}^{\lambda}_{\mu}$ be a basis of $H^0(G/B, \mathcal{L}_\lambda)_\mu$

~~There~~

$$\mathrm{Card}(\hat{G}^{\lambda}_{\mu}) = \mathrm{Card}(B(\lambda)_\mu) = \mathrm{Card}(\mathrm{Irr}(\overline{K^\vee t_\lambda K^\vee \cap U^- t_\mu K^\vee}))$$

By Gaussent-Littelmann ~~and~~ we know explictly the bijection

$$B(\lambda)_\mu \longleftrightarrow \mathrm{Irr}(K t_\lambda K \cap U^- t_\mu K)$$

Modular forms

Let $\Lambda_{\mathbb{Z}}$ be a lattice of rank $2g$ in $\mathbb{C}^g$.

An abelian variety of dimension g is $\mathbb{C}^g/\Lambda_{\mathbb{Z}}$ ~~with~~ which can be embedded into projective space.

An elliptic curve is an abelian variety with $g=1$.

A polarized abelian variety is a pair $(T, \mathcal{L})$ where

T is an abelian variety

$\mathcal{L}$ is an ample line bundle on T

Theta functions are elements of $H^0(T, \mathcal{L})$.

~~Let $d_1, \dots, d_g \in \mathbb{Z}_{>0}$ with $d_1 | d_2 | \dots | d_g$ and $\Delta = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_g \end{pmatrix}$~~

~~$Sp(\Delta, \mathbb{Z}) =$~~

There is a bijection

$$\left\{ \begin{matrix} \text{polarized abelian} \\ \text{varieties} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{polarized Hodge structures} \\ \text{of weight } 1 \end{matrix} \right\}$$

~~Let~~ The Siegel upper half plane of degree g is

$$\mathfrak{S}_g = \{ \tau \in M_g(\mathbb{C}) \mid \tau^t = \tau \text{ and } \operatorname{Im} \tau > 0 \}$$

$\mathfrak{S}_g$ is the period domain for polarized Hodge structures of weight 1.

$$\mathfrak{S}_g \simeq Sp(2g, \mathbb{R}) / K_{\mathbb{R}} \quad \text{where } K_{\mathbb{R}} = \{ ??? \}. \ (\text{a compact group})$$

Let $d_1, \dots, d_g \in \mathbb{Z}_{>0}$ with $d_1 | d_2 | \cdots | d_g$ and $\Delta = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_g \end{pmatrix}$

$$Sp(\Delta, \mathbb{Z}) = \left\{ M \in GL(2g, \mathbb{Z}) \;\middle|\; M \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} M^t = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \right\}$$

$$\Gamma_\Delta(n) = \{ M \in Sp(\Delta, \mathbb{Z}) \mid M \equiv I_{2g} \ (\mathrm{mod}\, n) \}$$

Then

$$Sp(\Delta, \mathbb{Z}) \backslash \mathfrak{S}_g = \left\{ \begin{array}{c} \text{polarized abelian varieties} \\ \text{of type } \Delta \end{array} \right\} = \mathcal{A}_\Delta$$

$$\Gamma_\Delta(n) \backslash \mathfrak{S}_g = \left\{ \begin{array}{l} \text{level } n \text{ polarized abelian} \\ \text{varieties of type } d_1, \dots d_g \end{array} \right\} = \mathcal{A}_\Delta(n)$$

<u>Modular forms</u> of level n and weight k are elements of

$$H^0(\mathcal{A}_g(n), \mathcal{L}^{\otimes k}).$$