What is a line bundle? informal working seminar 04.06.2010 ①
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A line bundle $\mathcal{L}$ is a locally trivial vector bundle of rank 1.

The corresponding principal $\mathbb{C}^\times$-bundle

Let $\begin{matrix} \mathcal{L} \\ \downarrow p \\ X \end{matrix}$ and $0 : X \to \mathcal{L}$ the 0-section.

Let $\mathcal{L}^+ = \{ l \in \mathcal{L} \mid l \notin \operatorname{im} 0 \}$ is a principal $\mathbb{C}^\times$-~~bu~~bundle.

Let $\mathbb{C}^\times$ act on $\mathcal{L}^+ \times \mathbb{C}$ by

$$\lambda(y, u) = (\lambda^{-1} y, \lambda u)$$

Then $\mathcal{L} = \dfrac{\mathcal{L}^+ \times \mathbb{C}}{\mathbb{C}^\times}$.

~~Let~~ Principal G-bundles Let G be a group.

Let $\begin{matrix} P \\ \downarrow p \\ X \end{matrix}$ be a principal G-bundle.

Let $\mathcal{U} = (U_i)$ be an open cover of X.

Let $s_i : U_i \to G$ be a section of P over U_i.

Let $g_{ij} : U_i \cap U_j \to G$ be given by

$$s_j = s_i g_{ij}$$

These satisfy $g_{ik} = g_{ij} g_{jk}$ on $U_i \cap U_j \cap U_k$

If $s_i' : U \to G$ is a different choice of sections then

$$s_i' = s_i h_i \quad \text{and} \quad g_{ij}' = h_i^{-1} g_{ij} h_j .$$

A <u>1-cocycle</u> is a family $g_{ij} : U_i \cap U_j \to G$ such that

$$g_{ik} = g_{ij} g_{jk} \quad \text{on} \quad U_i \cap U_j \cap U_k$$

Two one cocycles (g_{ij}) and (g_{ij}') are <u>cohomologous</u> (differ by a coboundary) if there exists a family $h_i : U_i \to G$ such that

$$g_{ij}' = h_i^{-1} g_{ij} h_j .$$

<u>Consider $\mathbb{P}^1$</u>

$$\mathbb{P} = \{ [x,y] \in \mathbb{C}^2 \setminus (0,0) \mid [\lambda x, \lambda y] = [x,y] \text{ for } x,y \in \mathbb{C} \text{ and } \lambda \in \mathbb{C}^\times \}$$

Then $[x,y] = [1, x^{-1}y]$ if $x \neq 0$, and

$[x,y] = [xy^{-1}, 1]$ if $y \neq 0$

and $U_1 = \{ [1,z] \mid z \in \mathbb{C} \}$ and $U_2 = \{ [z,1] \mid z \in \mathbb{C} \}$

form an open cover of $\mathbb{P}^1$ with

$$U_1 \cap U_2 = \{ [1,z] \mid z \in \mathbb{C}^\times \} \text{ with } [1,z] = [z^{-1}, 1].$$

Then $s_1 : U_1 \to \mathbb{C}^\times$ is a map $s_1 : \mathbb{C} \to \mathbb{C}^\times$

and $s_2 : U_2 \to \mathbb{C}^\times$ is a map $s_2 : \mathbb{C} \to \mathbb{C}^\times$.

and $g_{12} : U_1 \cap U_2 \to \mathbb{C}^\times$ is a map $\mathbb{C}^\times \to \mathbb{C}^\times$.

Line bundles on coset spaces

Claim $\mathbb{P}^1 = SL_2(\mathbb{C})/B$. where $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL_2(\mathbb{C}) \right\}$

$$= \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{C}) \right\}.$$

Well

$$GL_2(\mathbb{C}) = B \sqcup B s_1 B \quad \text{where } n_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $B n_1 B = \{ x_\alpha(c) n_1 B \mid c \in \mathbb{C} \}$ where $x_\alpha(c) = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$

Alternatively $GL_2(\mathbb{C}) = \{ x_{-\alpha}(c) B \mid c \in \mathbb{C} \} \cup n_1 B$

where $x_{-\alpha}(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$.

Furthermore

$$x_{-\alpha}(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1 & -c^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & 1 \\ 0 & -c^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} +c^{-1} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & +1 \\ & -c^{-1} \end{pmatrix} = \begin{pmatrix} +1 & 0 \\ +c & 1 \end{pmatrix}$$

and so $x_{-\alpha}(c) B = x_\alpha(-c^{-1}) n_1 B$.

Then a global section of line bundle is a function $f : SL_2(\mathbb{C}) \to \mathbb{C}$.

such that

$$f(gb) = f(g) \chi^k(b).$$

Constructing the line bundle from the cocycle

Let g_{ij} be a 1-cocycle ~~of~~ X with open cover $\mathcal{U}$ and values in $\mathcal{F} = \underline{\mathbb{C}}_X^\times$ (ie. $\mathcal{F}(U) = \{ f : U \to \mathbb{C}^\times \}$.

Define

$$\mathcal{L}^+ = \frac{\coprod U_i \times \mathbb{C}^\times}{\langle (x,\lambda)_i = (x, \lambda g_{ij}(x))_j \text{ for } x \in U_i \cap U_j \rangle}$$

Then global sections of $\mathcal{L}$ correspond to families $s_i : U_i \longrightarrow \mathbb{C}$ such that ~~$s_i = s_j$ on $U_i \cap U_j$~~.

~~and~~ ~~$\frac{s_i}{s_j} = g_{ij}$ on $U_i \cap U_j$~~

ie. $\quad s_j = s_i g_{ij}$ on $U_i \cap U_j$

Some line bundles on $\mathbb{P}^1$

$$\mathbb{P}^1 \cong SL_2(\mathbb{C})/B$$

$$[x,y] \longmapsto \begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix} B \quad \text{if } x \neq 0$$

$$[0,y] \longmapsto \begin{pmatrix} 0 & -y^{-1} \\ y & 0 \end{pmatrix} B$$

$SL_2(\mathbb{C})$ is generated by $x_\alpha(c) = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ and $x_{-\alpha}(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$

and

$$x_\alpha(c) x_{-\alpha}(-c^{-1}) x_\alpha(c) = n_\alpha(c) = \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix}$$

B is generated by $x_\alpha(c)$ and $h_{\alpha^\vee}(c)$.

Let $k \in \mathbb{Z}$ and let

~~Let~~ $\chi^k : B \longrightarrow \mathbb{C}^\times$ be given by

$\chi^k \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^k$. Let $\mathbb{C}_k = \mathrm{span}\{v\}$ with $\begin{pmatrix} a & b \\ a & a^{-1} \end{pmatrix} v = a^k v$.

Then define

$$\mathcal{L}_k = \frac{G \times_B \mathbb{C}_k}{\langle (gb, \lambda v) = (g, \lambda b v) \rangle} \xrightarrow{p} G/B$$

$$(g, \lambda v) \longmapsto gB$$

~~Then~~

Then, a global section of $\mathcal{L}_L$ is $s : G/B \longrightarrow \mathcal{L}_k$

~~$gB \mapsto (gb, s(gb)v)$~~

~~with the condition~~

~~$s(gb)$~~

is a function $s : G \longrightarrow \mathcal{L}_k$

$$g \longmapsto (g, s(g)v)$$

with the condition that

$$(gb, s(gb)v) = (g, s(gb) b v) = (g, s(gb) \chi^k(b) v)$$

$$= (g, s(g)v).$$

So $s(g) = s(gb)\chi^k(b)$ for all $b \in B$.

So

$s_1 : \{x_\alpha(c) n_\alpha B \mid c \in \mathbb{C}\} \longrightarrow \mathbb{C}$

$s_2 : \{x_{-\alpha}(c) B \mid c \in \mathbb{C}\} \longrightarrow \mathbb{C}$

but

$s_1(c) = s_2(-c^{-1}) c^k$.

I.e. $\frac{s_1}{s_2} = g_{12}$ where $g_{12} = c^k$