Combinatorics and spherical functions

The Weyl group W_0 acts on $\mathfrak{h}^*_{\mathbb{Z}}$

W_0 has generators $s_1, \ldots, s_n$ with relations

$$s_i^2 = 1 \quad \text{and} \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij}\text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}\text{ factors}}$$

where $\frac{\pi}{m_{ij}} = \mathfrak{h}^{\alpha_i^\vee} \measuredangle \mathfrak{h}^{\alpha_j^\vee}$. Let

$$u_0 = \sum_{w \in W_0} w \quad \text{and} \quad e_0 = \sum_{w \in W_0} (-1)^{\ell(w)} w \quad \text{in } \mathbb{C}W_0.$$

so that $w u_0 = u_0$ and $e_0 w = (-1)^{\ell(w)} w$.

The Hecke algebra H_0 has generators $T_1, \ldots, T_n$ with

$$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_i + 1 \quad \text{and} \quad \underbrace{T_i T_j T_i \cdots}_{m_{ij}\text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}\text{ factors}}.$$

Let $T_w = T_{i_1} \cdots T_{i_\ell}$ if $w = s_{i_1} \cdots s_{i_\ell}$ is reduced

$\{T_w \mid w \in W_0\}$ is a basis of H_0.

Let

$$\mathbb{1}_0 = \sum_{w \in W_0} (t^{\frac{1}{2}})^{\ell(w)} T_w \quad \text{and} \quad \varepsilon_0 = \sum_{w \in W_0} (-t^{-\frac{1}{2}})^{\ell(w)} T_w.$$

so that

$$T_w \mathbb{1}_0 = (t^{\frac{1}{2}})^{\ell(w)} \mathbb{1}_0 \quad \text{and} \quad \varepsilon_0 T_w = (-t^{-\frac{1}{2}})^{\ell(w)} \varepsilon_0.$$

Affine Hecke algebra H and Gindikin-Karpelevich.

$\mathbb{C}[X] = \mathrm{span}\{X^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*\}$ and $X^\lambda X^\mu = X^{\lambda+\mu}$.

H is generated by subalgebras H_0 and $\mathbb{C}[X]$ with

$$T_i X^\lambda = X^{s_i\lambda} T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X^{-\alpha_i}} \left(X^\lambda - X^{s_i\lambda}\right).$$

Intertwiners τ_i are $\tau_i X^\lambda = X^{s_i\lambda}\tau_i$,

$$\tau_i = T_i - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X^{-\alpha_i}} \quad \text{and} \quad T_i = \tau_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X^{-\alpha_i}}.$$

In this language Gindikin-Karpelevich is

$$\mathbf{1}_0 = \sum_{w \in W_0} \tau_w \prod_{\substack{\alpha \in R^+ \\ w\alpha \in R^-}} \left(t^{\frac{1}{2}}\right) \left(\frac{1 - t^{-1} X^{-\alpha}}{1 - X^{-\alpha}}\right)$$

$$\varepsilon_0 = \sum_{w \in W_0} \tau_w \prod_{\substack{\alpha \in R^+ \\ w\alpha \in R^-}} \left(-t^{-\frac{1}{2}}\right) \left(\frac{1 - t X^{-\alpha}}{1 - X^{-\alpha}}\right)$$

see Prop. 3.15 in the thesis of Martha Yip.

$\{T_w X^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*, w \in W_0\}$ is a basis of H.

Casselman-Shalika

Case $q=0=t=0$; Hermann Weyl; $G(\mathbb{C})$

$$u_0\mathbb{C}[X] = \mathbb{C}[X]^{W_0} \xrightarrow{\sim} \mathbb{C}[X]^{\det} = \varepsilon_0\mathbb{C}[X]$$

$$m_\lambda = u_0 X^\lambda$$

Weyl character = Schur function $= s_\lambda \longleftarrow\!\shortmid\; a_{\lambda+\rho} = \varepsilon_0 X^{\lambda+\rho}$

$h \longmapsto a_\rho h$

Weyl denominator = Vandermonde det.

Case $q=0$; Lusztig; $G(\mathbb{C}((t)))$

$$\mathbb{C}[X]^{W_0} = Z(H) \longrightarrow \mathbb{1}_0 H \mathbb{1}_0 \xrightarrow{\sim} \varepsilon_0 H \mathbb{1}_0$$

$$P_\lambda(0,t) \longleftarrow\!\shortmid\; \mathbb{1}_0 X^\lambda \mathbb{1}_0$$

$$s_\lambda \longleftarrow\!\shortmid\; C_\lambda \longleftarrow\!\shortmid\; A_{\lambda+\rho} = \varepsilon_0 X^{\lambda+\rho}\mathbb{1}_0$$

$$h \longmapsto A_\rho h$$

$H\mathbb{1}_0 = \mathbb{C}[X]\mathbb{1}_0$ = polynomial representation

$P_\lambda(0,t)$ = Hall-Littlewood poly = Macdonald Spherical function

$$A_\rho = \prod_{\alpha\in R^+}\left(t^{\frac12}X^{\alpha/2} - t^{-\frac12}X^{-\alpha/2}\right) = \left(t^{\frac12}\right)^{\ell(w_0)} X^\rho \prod_{\alpha\in R^+}\left(1-t^{-1}X^{-\alpha}\right)$$

(see arXiv 0401298 with Nelsen).

<u>Case</u> q,t; Cherednik-Opdam-Macdonald; $G(\mathbb{C}((t,s)))$?

Let $\tilde{H}$ be the double affine Hecke algebra

$\{X^\mu T_w Y^{\lambda^\vee} \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}}, w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$ is a basis of $\tilde{H}$.

Let $\mathbb{1}$ be such that $T_w \mathbb{1} = (t^{\frac{1}{2}})^{\ell(w)} \mathbb{1}$ and $Y^{\lambda^\vee}\mathbb{1} = t^{\langle \lambda^\vee, \rho \rangle}\mathbb{1}$.

$$\mathbb{C}[X]^{W_0}\mathbb{1} = \mathbb{1}_0 \tilde{H} \mathbb{1} \qquad\qquad \varepsilon_0 \tilde{H}\mathbb{1}$$

$$P_\lambda(q,t)\mathbb{1} \longleftarrow \mathbb{1}_0 E_\lambda \mathbb{1}$$

$$P_\lambda(q,qt)\mathbb{1} \longleftarrow A_{\lambda+\rho}(q,t) = \varepsilon_0 E_{\lambda+\rho}\mathbb{1}$$

$$h \longmapsto A_\rho h$$

$\tilde{H}\mathbb{1} = \mathbb{C}[X]\mathbb{1}$ = polynomial representation of $\tilde{H}$

$E_\lambda = E_\lambda(q,t)$ = nonsymmetric Macdonald polynomial

$A_\rho = \prod_{\alpha \in R^+} (t^{\frac{1}{2}} X^{\alpha/2} - t^{-\frac{1}{2}} X^{-\alpha/2})$, see Prop 2.13 in Yip's thesis.

(see Macdonald Séminaire Bourbaki 1995).

Writing Whittaker vectors & HH in terms of crystals

(A) arXiv 0601343

I know {alcovewalks} = crystal, from Littelmann

I showed alcove walks $\Longleftrightarrow$ affine Hecke algebra

following Schwer.

(B) arXiv 0801.0709 with Parkinson-Schwer

We show; extending Gaussent-Littelmann from spherical to Iwahori;

$$\left\{\begin{matrix}\text{labeled alcove}\\ \text{walks}\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{points of}\\ IwI \cap U^- vI\end{matrix}\right\}$$

where

$$\begin{array}{ccc} G = G(\mathbb{C}((t))) & & \\ \cup\!\!\mid & & \\ K = G(\mathbb{C}[[t]]) & \xrightarrow[\Phi]{t=0} & G(\mathbb{C}) \\ \cup\!\!\mid & & \cup\!\!\mid \\ I = \Phi^{-1}(B) & \longrightarrow & B \end{array}$$

and

$$G = \bigsqcup_{w \in W_0 \ltimes \mathfrak{h}_{\mathbb{Z}}} IwI \quad \text{and} \quad G = \bigsqcup_{v \in W_0 \ltimes \mathfrak{h}_{\mathbb{Z}}} U^- vI$$

In the non-metaplectic case

$$T_i = t^{\frac{1}{2}} s_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X^{-\alpha_i}} (1 - s_i) \quad \text{as operators on } \mathbb{C}[X]$$

providing an alcovewalk = crystal interpretation of Chinta-Gunnells.