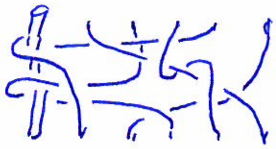
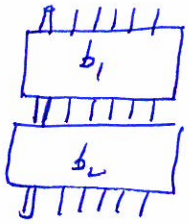
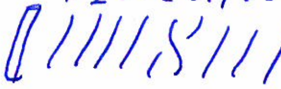
On affine BMW algebras, Univ. of Adelaide
Pure Maths Seminar 25.06.2010

The affine braid group $\mathcal{B}_k$

$b =$  $\in \mathcal{B}_5$

Product: $b_1 b_2 =$ 

$\mathcal{B}_k$ is given by generators

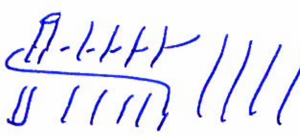
$X^{\varepsilon_1} =$  and $T_i =$ 

, $i = 1, \ldots, k-1$.

with relations

$$X^{\varepsilon_1} T_1 X^{\varepsilon_1} T_1 = T_1 X^{\varepsilon_1} T_1 X^{\varepsilon_1},$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{and} \quad T_i T_j = T_j T_i \text{ if } j \neq i \pm 1.$$

Let $X^{\varepsilon_i} =$  for $i = 1, 2, \ldots, k$

Then $X^{\varepsilon_i} X^{\varepsilon_j} = X^{\varepsilon_j} X^{\varepsilon_i}$ and

$\mathbb{C}[X^{\pm\varepsilon_1}, \ldots, X^{\pm\varepsilon_k}]$ is a subalgebra of $\mathbb{C}\mathcal{B}_k$.

The affine BMW algebra W_k

Let $z, q, L_1, \ldots, L_k \in \mathbb{C}^\times$ or let $C = \mathbb{C}[z^{\pm 1}, q^{\pm 1}, L_1^{\pm 1}, \ldots, L_k^{\pm 1}]$.

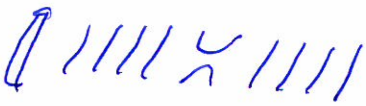
Define $z_1^{(\ell)} \in C$ by

$$z_1(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} z_1^{(\ell)} u^{-\ell}$$

$$= \frac{z^{+1}}{(q-q^{-1})} \frac{(1-u^{-1}z^{-1})}{(1-zu^{-1})} \frac{(u-q)}{(u-1)} \frac{(u+q^{-1})}{(u+1)} \prod_{i=1}^{r} \frac{(u-qL_i)(u-qL_i^{-1})}{(u-q^{-1}L_i)(u-q^{-1}L_i^{-1})}$$

Let $Y_1, \ldots, Y_k$ and $E_1, \ldots, E_{k-1}$ in CB_k be given by

$$Y_i = zX^{\varepsilon_i} \quad \text{and} \quad T_iY_i = Y_{i+1}T_i + (q-q^{-1})(1-E_i)$$

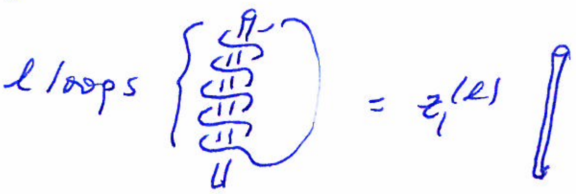
Pictorially, $E_i =$ 

The *affine* *BMW* *algebra* W_k is the quotient of CB_k by

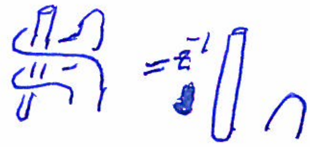
$$T_i^{\pm 1}E_i = E_iT_i^{\pm 1} = z^{\mp 1}E_i,$$

$$E_iT_{i+1}^{\pm 1}E_i = E_iT_{i-1}^{\pm 1}E_i = z^{\pm 1}E_i,$$

$$E_1Y_1^{\ell}E_1 = z_1^{(\ell)}E_1$$



$$E_iY_iY_{i+1} = Y_iY_{i+1}E_i = E_i$$



<u>The quantum group</u> $U_q\mathfrak{g}$, $\mathfrak{g}=\mathfrak{sp}_{2r}$

$Sp_{2r}(\mathbb{C})$ has Lie algebra $\mathfrak{sp}_{2r}$
which has enveloping algebra $U\mathfrak{g}$
with corresponding quantum group $U_q\mathfrak{g}$.

<u>Moral</u>: For (most) representation theory
$Sp_{2r}(\mathbb{C})$, $\mathfrak{sp}_{2r}$, $U\mathfrak{g}$, $U_q\mathfrak{g}$ are "the same"

Harish-Chandra ... Joseph-Letzter ... say

$$Z(U_q\mathfrak{g}) \cong \mathbb{C}[L_1^{\pm1},\dots,L_n^{\pm1}]^{W_0}$$

where W_0 is the Weyl group of Sp_{2r}

<u>Moral</u>: $z_1^{(t)} \in Z(U_q\mathfrak{g})$

$U_q\mathfrak{g}$ has a "standard" $2r$ dimensional representation
V with basis $\{b_{\varepsilon_1},\dots,b_{\varepsilon_r},b_{-\varepsilon_r},\dots,b_{\varepsilon_1}\}$ $\left(\begin{array}{l}Sp_{2r}(\mathbb{C}) \text{ acts}\\ \text{on } \mathbb{C}^r\end{array}\right)$

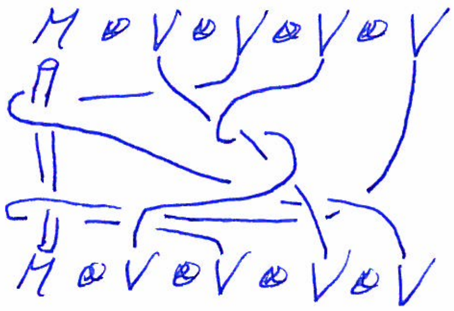
Let
M be (almost) any other $U_q\mathfrak{g}$ module.

Schur-Weyl duality

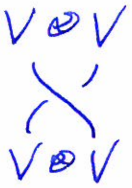
Theorem Let $z = q^{2r+1}$. The affine BMW algebra W_k acts on $M \otimes \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ factors}}$ by

$$p(m \otimes v_1 \otimes \cdots \otimes v_k) = pm \otimes v_1 \otimes \cdots \otimes v_k, \quad \text{if } p \in Z(U_q\mathfrak{g})$$

and

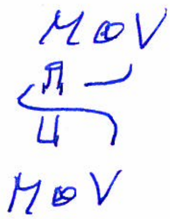


where



$$v_1 \otimes v_2 \longmapsto \sum_k b_k v_2 \otimes a_k v_1$$

and



$$m \otimes v \longmapsto \sum_{k,\ell} b_k a_\ell m \otimes a_k b_\ell v$$

where $R = \sum_k a_k \otimes b_k$ is the universal R-matrix of $U_q\mathfrak{g}$

So $\Phi : W_k \longrightarrow \mathrm{End}_{U_q\mathfrak{g}}(M \otimes V^{\otimes k})$

and in many cases, as a $(U_q\mathfrak{g}, W_k)$-bimodule

$$M \otimes V^{\otimes k} \cong \bigoplus_\lambda L(\lambda) \otimes W_k^\lambda$$

$L(\lambda)$: $U_q\mathfrak{g}$-module

W_k^λ: irreducible W_k-module

The Kazhdan-Lusztig conjecture

$$M(\lambda) = \mathrm{Ind}_{U_q\mathfrak{b}}^{U_q\mathfrak{g}}(\mathbb{C}_\lambda)$$

where $\mathbb{C}_\lambda$ is a onedimensional $U_q\mathfrak{b}$-module.

Question: (BGG-Verma) What are the composition factors of $M(\lambda)$ and what are their multiplicities.

Kazhdan-Lusztig conjectured a "combinatorial" method of computing these factors and multiplicities.

Beilinson-Bernstein and Brylinski-Kashiwara proved it using

flag varieties, perverse sheaves, Weil conjectures.

$M(\lambda)$ has a (unique up to normalization)

bilinear form $\langle\ \rangle_\lambda : M(\lambda)\times M(\lambda) \to \mathbb{C}$

such that $\langle xm_1, m_2\rangle = \langle m_1, x^t m_2\rangle$

where
$$\begin{aligned} U_q\mathfrak{g} &\to U_q\mathfrak{g} \\ x &\mapsto x^t \end{aligned}$$
is the antiautomorphism

coming from transpose of matrices.

Ideas

Idea 1 of Jantzen: Generic versus singular behaviour

The module $M(\lambda)$ and the form $\langle\ \rangle_\lambda$

mostly behave well in λ,

i.e. $\langle\ \rangle : M \times M \longrightarrow \mathbb{C}[L_1^{\pm 1}, \ldots, L_n^{\pm 1}]$

with $M(\lambda)$ is M evaluated at λ

and $\langle\ \rangle_\lambda$ is $\langle\ \rangle$ evaluated at λ.

Idea 2 of Jantzen: Translation

Suppose we know everything about

$$\langle\ \rangle_\lambda : M(\lambda) \times M(\lambda) \longrightarrow \mathbb{C}$$

for some specific λ. Define

$$\langle\ \rangle_\otimes : (M(\lambda) \otimes V) \times (M(\lambda) \otimes V) \longrightarrow \mathbb{C} \quad \text{by}$$

$$\langle m_1 \otimes v_1, m_2 \otimes v_2 \rangle_\otimes = \langle m_1, m_2 \rangle_\lambda \langle v_1, v_2 \rangle.$$

Does this give us information about

$$\langle\ \rangle_{\lambda+shift} : M(\lambda+shift) \times M(\lambda+shift) \longrightarrow \mathbb{C}.$$

Our idea: Use affine BMW algebra to compute $\langle\ \rangle_\otimes$ and $\langle\ \rangle_{\lambda+shift}$ for $U\mathfrak{g}$ of $Sp_{2r}(\mathbb{C})$

Theorem It works!