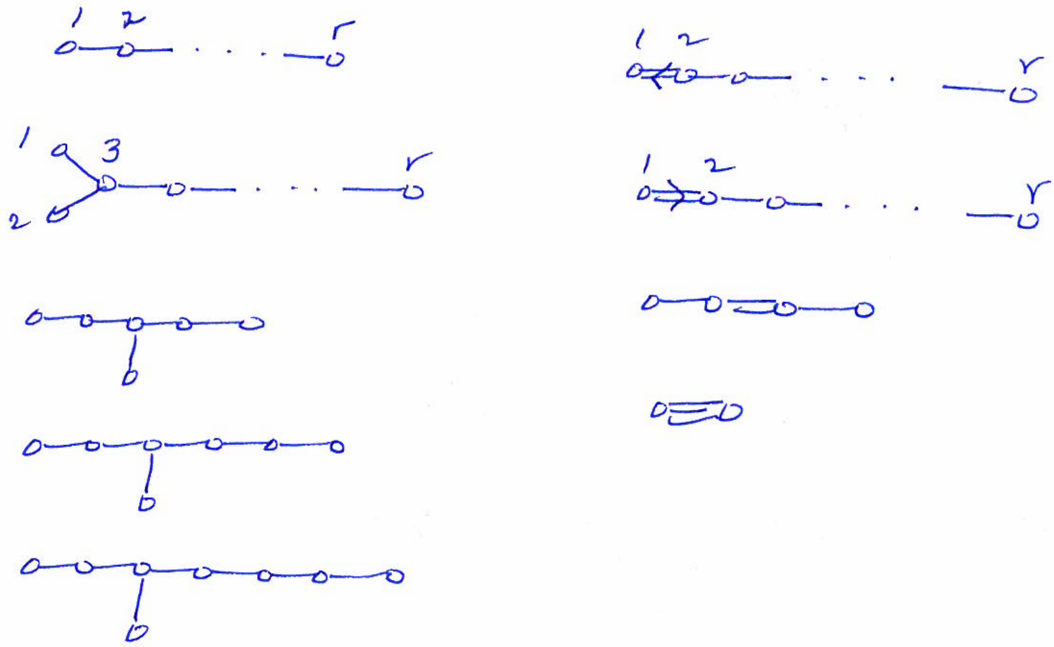
The glass bead game, Colloquium, Univ. of Adelaide ①

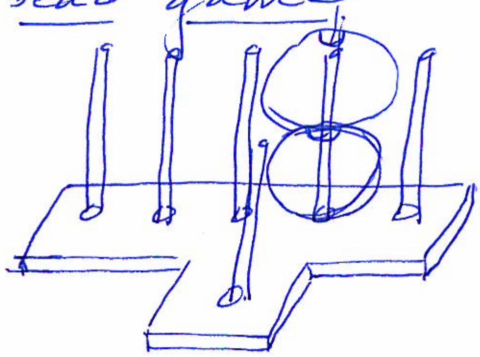
25.06.2010

The Glass Bead games

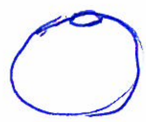
The finite dimensional complex simple Lie algebras are in bijection with



The glass bead game



Board

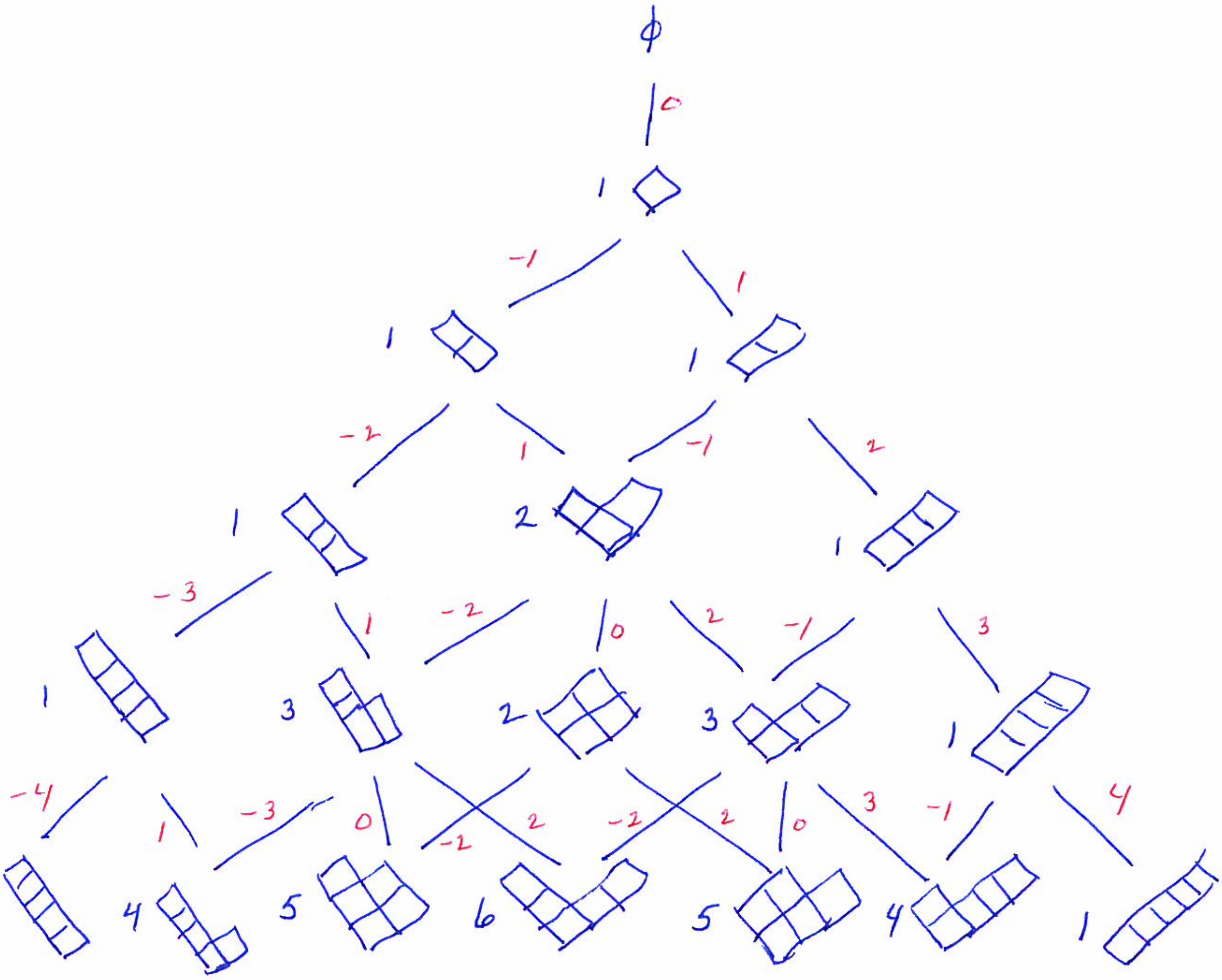


Beads

Standard tableaux

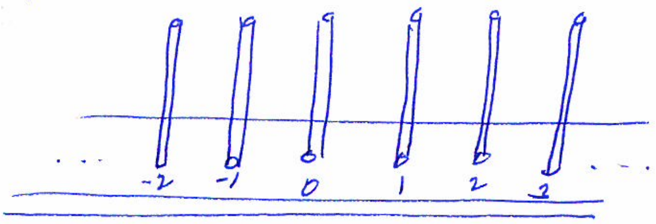
A *shape* λ is a configuration of beads

A *standard tableau* of shape λ is a sequence $i_1, \ldots, i_k$ of runners such that playing the sequence $i_1, \ldots, i_k$ results in the shape λ.



Standard tableaux of shape λ are paths to λ.

In this example our board is



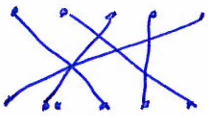
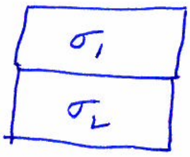
Algebras

An *algebra* is a vector space A with a product such that A is a ring with identity.

Artin-Wedderburn Theorem Any decent algebra is isomorphic to a direct sum of matrix algebras

$$A \cong M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_r}(\mathbb{C}) = \left\{ \begin{pmatrix} \underbrace{\square}_{d_1} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\square}_{d_r} \end{pmatrix} \right\}$$

The symmetric group S_k

 $\in S_5$ product:  $= \sigma_1 \sigma_2$

$\mathbb{C}S_k = \text{span}\{\sigma \mid \sigma \in S_k\}$ is an algebra with

$$\dim(\mathbb{C}S_k) = k!$$

Generators: $s_i = \; |\,|\,|\,|\underset{i\;\; i+1}{X}|\,|\,|$, $\quad i = 1, 2, \ldots, k-1$

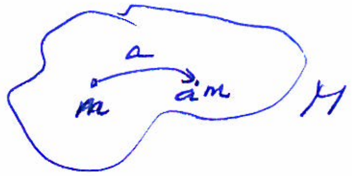
Relations: $s_i^2 = 1$, $\quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

and $s_i s_j = s_j s_i$ if $j \neq i \pm 1$.

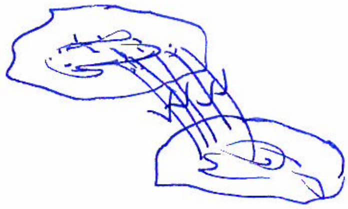
Modules = Representation Theory

Let A be an algebra.

An <u>A-module</u> is a vector space M with an action of A on M



An A-module M is <u>irreducible</u> if M has no submodules except M and 0.



N is a submodule.

<u>Theorem</u> Define a vector space

$$S^\lambda = \mathrm{span}\left\{ v_{i_1\cdots i_k} \,\middle|\, \begin{array}{c}(i_1,\ldots,i_k) \text{ is a standard tableau} \\ \text{of shape } \lambda\end{array} \right\}$$

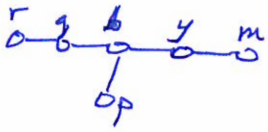
and an S_k-action on S^λ by

$$s_j v_{i_1\cdots i_k} = \frac{1}{i_j - i_{j+1}} v_{i_1\cdots i_k} + \left(1 + \frac{1}{i_j - i_{j+1}}\right) v_{i_1\cdots i_{j+1} i_j \cdots i_k}$$

with $v_{i_1\cdots i_{j+1} i_j \cdots i_k} = 0$ if $i_1\cdots i_{j+1} i_j \cdots i_k$ is not a standard tableau of shape λ.

The S^λ are irreducible S_k-modules.

Quiver Hecke algebras

Let $Q =$ 

and $k \in \mathbb{Z}_{>0}$.

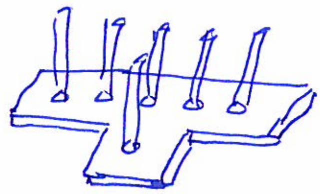
The quiver Hecke algebra R_k is given by

Generators: $e(i_1, \dots, i_k)$, $y_1, \dots, y_k$, $\psi_1, \dots, \psi_{k-1}$

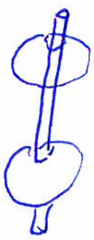
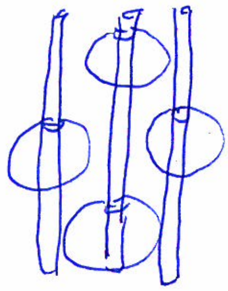
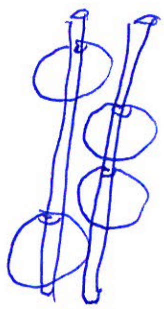
Relations:

and $\deg(e(i_1, \dots, i_k)) = 0$, $\deg(y_j e(i_1 \cdots i_k)) = ?$ and $\deg(\cdots) = ?$

R_k is a graded algebra: $R_k = \bigoplus_{d \in \mathbb{Z}} R_k[d]$.

Let λ be a skew shape for 

i.e. any two beads on the same runner are separated by at least two beads:

if  then  or 

Define a vector space

$$R^\lambda = \mathrm{span}\left\{ v_{i_1\cdots i_k} \,\middle|\, \begin{array}{l}(i_1,\ldots,i_k) \text{ is a standard tableau} \\ \text{of shape } \lambda\end{array} \right\}$$

and define an R_k action on R^λ by

$$e(j_1,\ldots,j_k)v_{i_1\cdots i_k} = \delta_{i_1\cdots i_k, j_1\cdots j_k}, \qquad y_j v_{i_1\cdots i_k} = 0$$

$$\psi_j \cdot v_{i_1\cdots i_k} = v_{i_1\cdots i_{j+1} i_j \cdots i_k} \quad \text{and} \quad \deg(v_{i_1\cdots i_k}) = 0.$$

Then (Kleshchev-R) R^λ is an irreducible R-module.

~~Zoom out~~ Categorification

Let f_i be symbols. Let M be an R_k-module.

$$M = \bigoplus_{i_1,\ldots,i_k} \bigoplus_{d\in\mathbb{Z}} e(i_1\cdots i_k)M[d] \quad \left(\begin{array}{l}\text{as a vector}\\ \text{space}\end{array}\right)$$

and

$$\mathrm{char}\, M = \sum_{i_1,\ldots,i_k} \sum_{d\in\mathbb{Z}} q^d f_{i_1}\cdots f_{i_k} \dim(e(i_1\cdots i_k)M[d])$$

is the character of M (dimension generating function).

$$\{R_k\text{-modules}\} \longrightarrow U_q\mathfrak{g}$$

$$M \longmapsto \mathrm{char}(M)$$

Categorification

G is the Lie group corresponding to 

(a group that is also a space = manifold)

$\mathfrak{g}$ is the Lie algebra of G
(local structure of G)

$U\mathfrak{g}$ is the smallest algebra containing $\mathfrak{g}$

$U_q\mathfrak{g}$ = the quantum group
(the unique deformation = continuous family
of algebras containing $U\mathfrak{g}$)

R_α-modules $\xrightarrow{\text{Khovanov Lauda Rouquier}}$ $U_q\mathfrak{g}$ $\xleftrightarrow{\text{Ringel-Lusztig}}$ geometry of the quiver variety

irreducible modules $\longmapsto$ canonical basis $\longleftrightarrow$ irreducible pieces

standard basis $\longleftrightarrow$ naive pieces

$M \longmapsto \mathrm{char}(M)$

Varieties are unions of "irreducible pieces".

If $X \xrightarrow{f} Y$ is a morphism of varieties

then pieces of X have an image on Y: "naive pieces".