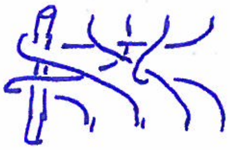
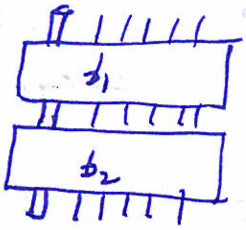
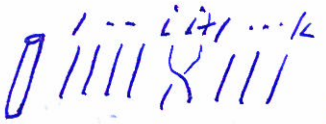
The center of the affine BMW algebra ICRT V Aug 9-14, 2000
Xi'an China

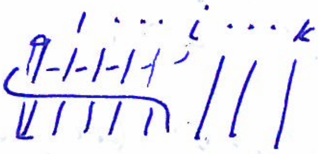
①

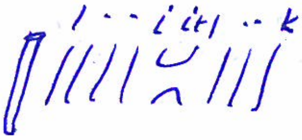
The affine braid group $\mathcal{B}_k$

$b =$  $\in \mathcal{B}_4$ and $b_1 b_2 =$ 

Then let q, z be constants and

$T_i =$ 

, $Y_i = z$ 

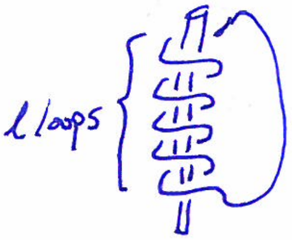
$E_i =$ 

with $T_iY_i = Y_{i+1}T_i - (q-q^{-1})Y_{i+1}(1-E_i)$.

The affine BMW algebras W_k are $\mathbb{C}\mathcal{B}_k$ with

 $= z^{-1}$ ,  $= z$ ,  $=$ ,  $=$ 

and



$= Z_1^{(l)}$ , where $Z_1^{(l)}$, $l \in \mathbb{Z}$, are constants

Remark: For some choices of $Z_1^{(l)}$, W_k is the 0 algebra.

The affine Hecke algebra H_k is W_k with

$E_i = 0$.

The <u>cyclotomic BMW algebras</u> are W_k with

$$(y_1-u_0)(y_1-u_1)\cdots(y_1-u_{r-1})=0$$

where $u_0, \dots, u_{r-1}$ are constants.

Note: $y_iy_j = y_jy_i$ for $1 \le i,j \le k$ and

$\mathbb{C}[y_1^{\pm1}, \dots y_k^{\pm1}]$ is a subalgebra of W_k.

<u>Theorem</u>

(a) (Bernstein-Zelevinsky)

$$Z(H_k) = \mathbb{C}[y_1^{\pm1}, \dots, y_k^{\pm1}]^{S_k}$$

where the symmetric group S_k acts by permuting $y_1, \dots, y_k$.

(b) (Daugherty-Ram-Virk)

$$Z(W_k) = \left\{ \begin{array}{l} p(y_1, \dots y_k) \in \mathbb{C}[y_1^{\pm1}, \dots, y_k^{\pm1}]^{S_k} \text{ such that} \\ p(y_1, y_1^{-1}, y_2, \dots, y_k) = p(y, 1, y_3, \dots, y_k) \end{array} \right\}$$

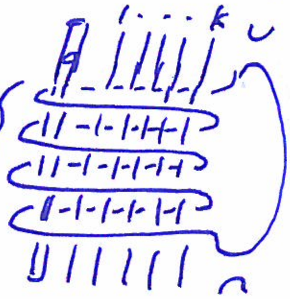
<u>Example</u> For $r \in \mathbb{Z}_{>0}$

$$p_r = y_1^r + y_2^r + \dots + y_k^r - (y_1^{-r} + y_2^{-r} + \dots + y_k^{-r}) \in Z(W_k).$$

Nazarov-Beliakova-Blanchet

Define $Z_{k+1}^{(\ell)} \in Z(W_k)$ by

$\boxed{Z_k^{(\ell)}}\ \overset{\cup}{\cap}$ = ℓ loops



$= E_{k+1} y_{k+1}^{\ell} E_{k+1}$, for $\ell \in \mathbb{Z}$.

Let

$$Z_{k+1}^{+}(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} Z_{k+1}^{(\ell)} u^{-\ell} \quad \text{and} \quad Z_{k+1}^{-}(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} Z_{k+1}^{(-\ell)} u^{-\ell}$$

<u>Theorem</u> (N-B-B)

(a) $$\left(Z_1^{+}(u) - \frac{z}{q-q^{-1}} - \frac{u^2}{u^2-1}\right)\left(Z_1^{-}(u) + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}\right)$$

$$= \frac{(u^2-q^2)(u^2-q^{-2})}{(u^2-1)^2(q-q^{-1})^2}$$

(b) $$Z_{k+1}^{+}(u) - \frac{z}{q-q^{-1}} - \frac{u^2}{u^2-1}$$

$$= \left(Z_1^{+}(u) - \frac{z}{q-q^{-1}} - \frac{u^2}{u^2-1}\right) \prod_{i=1}^{k} \frac{(u-y_i)^2(u-q^2 y_i^{-1})(u-q^{-2}y_i^{-1})}{(u-y_i^{-1})^2(u-q^2 y_i)(u-q^{-2}y_i)}$$

What are the $Z_1^{(\ell)}$?

Schur-Weyl duality

Let $\mathfrak{g} = \mathfrak{so}_{2r+1}, \mathfrak{sp}_{2r}$ and $\mathfrak{so}_{2r}$ and

$U_q\mathfrak{g}$ the corresponding quantum group.

Let $z = \epsilon q^y$ where

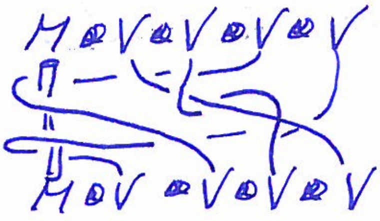
$$\epsilon = \begin{cases}1\\-1\\1\end{cases} \text{ and } y = \begin{cases}2r\\2r+1\\2r-1\end{cases} \text{ if } \mathfrak{g} = \begin{cases}\mathfrak{so}_{2r+1}\\ \mathfrak{sp}_{2r}\\ \mathfrak{so}_{2r}\end{cases}$$

Let V be the $U_q\mathfrak{g}$-module corresponding to the defining matrix representation of $\mathfrak{g}$.

If M and N are $U_q\mathfrak{g}$-modules then

$$\check{R}_{MN} : M \otimes N \xrightarrow{\sim} N \otimes M$$

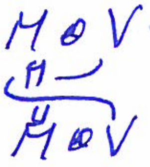
Let M be any $U_q\mathfrak{g}$-module. Then W_k acts on $M \otimes V^{\otimes k}$ by



where

$= \check{R}_{VV}$

and



$= \check{R}_{VM}\check{R}_{MV}$. Define functors

$$\{U_q\mathfrak{g}\text{-modules}\} \xrightarrow{F_\lambda} \{W_k\text{-modules}\}$$
$$M \longmapsto (M \otimes V^{\otimes k})^+_\lambda$$

where

$(M \otimes V^{\otimes k})^+_\lambda = \{$highest weight vectors of weight λ in $M \otimes V^{\otimes k}\}$

The ring $\mathbb{C}$

If M is a simple $U_q\mathfrak{g}$-module

$z_1^{(\ell)}$ =

acts on M by a constant.

So $z_1^{(\ell)} \in \mathbb{C}$ where $\mathbb{C} = Z(U_q\mathfrak{g})$!!

Now $U_q\mathfrak{g} = U^+U^0U^-$ with

$$U^0 = \mathbb{C}[L_1^{\pm1},\dots,L_r^{\pm1}]$$

where

$L_i v_{\varepsilon_i} = q v_{\varepsilon_i}$ and $L_i v_{\varepsilon_j} = v_{\varepsilon_j}$ if $j \neq i$

for a weight basis of V

$\{v_{\varepsilon_1},\dots,v_{\varepsilon_r},v_0,v_{-\varepsilon_r},\dots,v_{-\varepsilon_1}\}$, if $\mathfrak{g}=\mathfrak{so}_{2r+1}$

$\{v_{\varepsilon_1},\dots,v_{\varepsilon_r},v_{-\varepsilon_r},\dots,v_{-\varepsilon_1}\}$, if $\mathfrak{g}=\mathfrak{sp}_{2r}$

Theorem (Harish-Chandra)

$$Z(U_q\mathfrak{g}) = \mathbb{C}[L_1^{\pm1},\dots,L_r^{\pm1}]^{W_0}$$

where W_0 is the Weyl group of $\mathfrak{g}$.

What are $z_1^{(\ell)}$??

Theorem (Daugherty-Ram-Virk)

Let

$$Z_1^+(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} Z_1^{(\ell)} u^{-\ell} \quad \text{and} \quad Z_1^-(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} Z_1^{(-\ell)} u^{-\ell}$$

Then

$$Z_1^+(u) - \frac{z}{q-q^{-1}} - \frac{u^2}{u^2-1}$$

$$= \left(\frac{z}{q-q^{-1}}\right)\left(\frac{u-z^{-1}}{u-z}\right)\frac{(u+q)(u-q^{-1})}{(u+1)(u-1)} \prod_{i=1}^{r} \frac{(u-qL_i)(u-qL_i^{-1})}{(u-q^{-1}L_i)(u-q^{-1}L_i^{-1})}$$

and

$$Z_i^-(u) + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}$$

$$= \left(\frac{-z^{-1}}{q-q^{-1}}\right)\left(\frac{u-z}{u-z^{-1}}\right)\frac{(u+q^{-1})(u-q)}{(u+1)(u-1)} \prod_{i=1}^{k} \frac{(u-q^{-1}L_i^{+1})(u-q^{-1}L_i^{-1})}{(u-qL_i)(u-q^{-1}L_i^{-1})}.$$

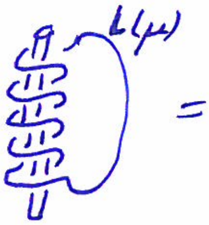
Proof): N-B-B compute the action of $Z_{k+1}^+(u)$ on irreducible W_k-modules. Use this and

$$V^{\otimes k} \cong \bigoplus_{\lambda} \underset{\substack{\text{irreducible} \\ U_q\mathfrak{g}\text{-module}}}{L(\lambda)} \otimes \underset{\text{irreducible } W_k\text{-module}}{W_k^{\lambda}}$$

to prove that our formulas are equivalent to NBB.

Proof 2 Fadeev-Takhtajan-Drinfeld-Reshetikhin-Baumann study central elements

$$z^{(\ell)}_{L(\mu)} =$$



$$= (id \otimes qtr_{L(\mu)})(R_{21}R)^{\ell})$$

where R is the R-matrix of $U_q\mathfrak{g}$ and qtr is the quantum trace

By Turaev-Wenzl

$$Z(U_q\mathfrak{g}) \longrightarrow \mathbb{C}[L_1^{\pm1},\dots,L_r^{\pm1}]^{W_0}$$

$$z^{(1)}_{L(\mu)} \longleftrightarrow s_\mu$$ where s_μ is the Weyl character

and define $s^{(\ell)}_\mu$ by

$$z^{(\ell)}_{L(\mu)} \longleftrightarrow s^{(\ell)}_\mu$$

Use The Baumann theorem

$$\sum_{w\in W_0} q^{\ell\langle w\lambda,\rho\rangle} s^{(\ell)}_{w\lambda} = \sum_{w\in W_0} q^{\langle w\lambda,\rho\rangle} s_{w\ell\lambda}$$

and some generating function identities from Halverson-Ram 1995.