Symmetry and identities (LHS = RHS).

Sum = Product

The Riemann zeta function: Let $s \in \mathbb{C}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\substack{p \in \mathbb{Z}_{>0} \\ p \text{ prime}}} \frac{1}{1-p^{-s}}$$

The Vandermonde

$$x_1^2 x_2 - x_2^2 x_1 - x_1^2 x_3 + x_2^2 x_3 + x_3^2 x_1 - x_3^2 x_2$$

$$= (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

$$x_1^3 x_2^2 x_3^1 x_4^0 - x_1^3 x_3^2 x_2^1 x_4^0 - x_1^3 x_2^2 x_4^1 x_3^0 + x_1^3 x_3^2 x_4^1 x_2^0 + x_1^3 x_4^2 x_2^1 x_3^0 - x_1^3 x_4^2 x_3^1 x_2^0$$
$$- x_2^3 x_1^2 x_3^1 x_4^0 + x_2^3 x_3^2 x_1^1 x_4^0 + x_2^3 x_1^2 x_4^1 x_3^0 - x_2^3 x_3^2 x_4^1 x_1^0 - x_2^3 x_4^2 x_1^1 x_3^0 + x_2^3 x_4^2 x_3^1 x_1^0$$
$$- x_3^3 x_1^2 x_2^1 x_4^0 + x_3^3 x_2^2 x_1^1 x_4^0 + x_3^3 x_1^2 x_4^1 x_2^0 - x_3^3 x_2^2 x_4^1 x_1^0 - x_3^3 x_4^2 x_1^1 x_2^0 + x_3^3 x_4^2 x_2^1 x_1^0$$
$$- x_4^3 x_1^2 x_2^1 x_3^0 + x_4^3 x_2^2 x_1^1 x_3^0 + x_4^3 x_1^2 x_3^1 x_2^0 - x_4^3 x_2^2 x_3^1 x_1^0 - x_4^3 x_3^2 x_1^1 x_2^0 + x_4^3 x_3^2 x_2^1 x_1^0$$

$$= (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$$

or

$$\sum_{w \in S_n} \det(w)\, x_{w(1)}^{n-1} x_{w(2)}^{n-2} \cdots x_{w(n)}^{0} = \prod_{1 \le i < j \le n} (x_i - x_j)$$

or

$$\det \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix} = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

$$\det\begin{pmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \\ x_4^3 & x_4^2 & x_4 & 1 \end{pmatrix} = (x_1-x_2)(x_1-x_3)(x_1-x_4) \cdot (x_2-x_3)(x_2-x_4) \cdot (x_3-x_4)$$

The <u>symmetric group</u> is

$S_n = \left\{ \begin{array}{l} n\times n \text{ matrices with} \\ \text{exactly one nonzero entry in each row and each col.} \\ \text{and nonzero entries } 1 \end{array} \right\}$

with operation matrix multiplication,
so that

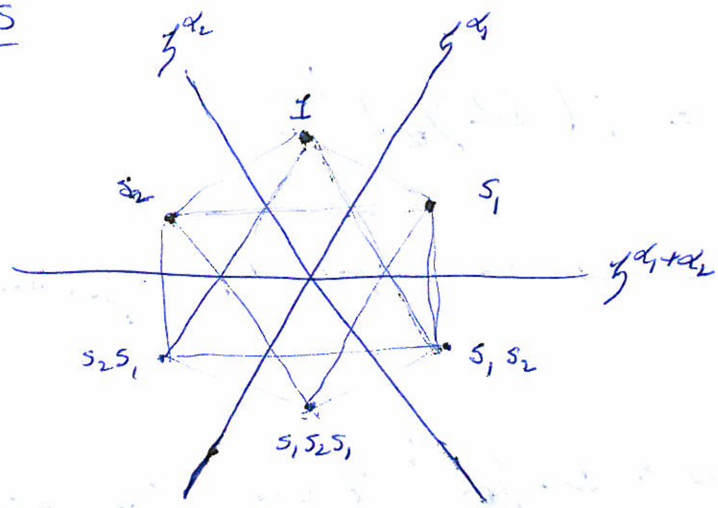
$$S_3 = \left\{ \begin{array}{l} \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix} \\ \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix} \end{array} \right\}$$

$= \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$ with $s_1s_2s_1 = s_2s_1s_2$
and $s_1^2 = 1$ and $s_2^2 = 1$

if

$$1 = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}$$

Symmetries



$$\sum_{\text{chambers } w} (-1)^{(\#\text{ of hyp between } w \text{ and } 1)} e^{-(\text{hyps between } w \text{ and } 1)}$$

$$= \prod_{\text{hyperplanes } \mathfrak{h}^{\alpha}} (1-e^{-\alpha}).$$

i.e.

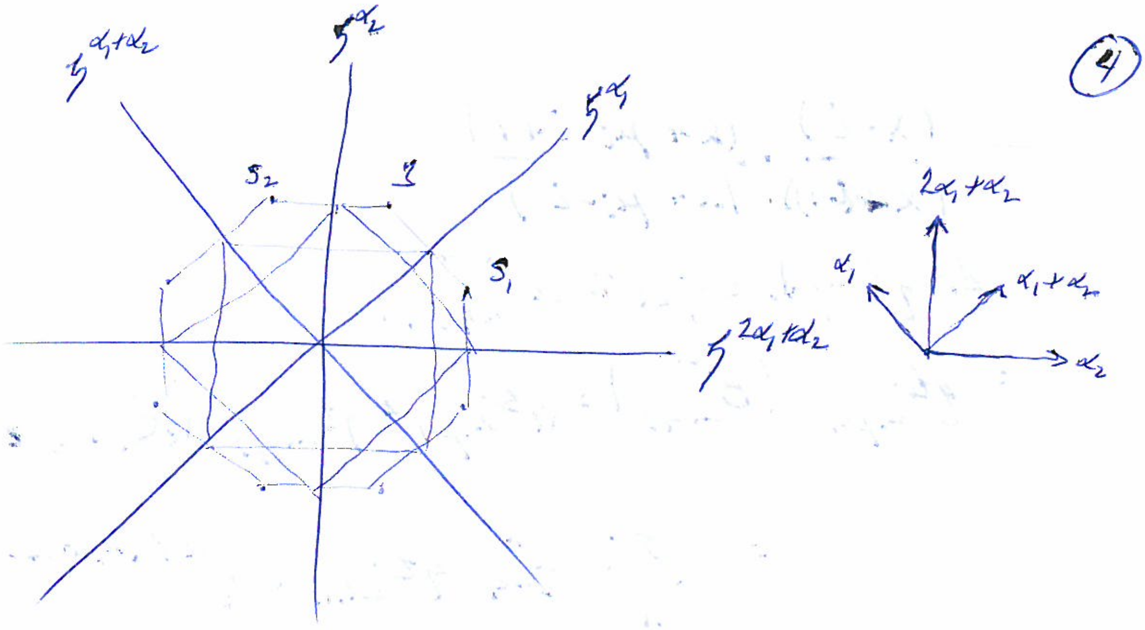
$$(-1)^0 e^0 + (-1)^1 e^{-\alpha_1} + (-1)^1 e^{-\alpha_2} + (-1)^2 e^{-\alpha_1-(\alpha_1+\alpha_2)}$$
$$+ (-1)^2 e^{-\alpha_2-(\alpha_1+\alpha_2)} + (-1)^3 e^{-\alpha_1-\alpha_2-(\alpha_1+\alpha_2)}$$
$$= (1-e^{-\alpha_1})(1-e^{-\alpha_2})(1-e^{-(\alpha_1+\alpha_2)}).$$

Do a variable change: $e^{-\alpha_1} = x_2 x_1^{-1}$, $e^{-\alpha_2} = x_3 x_2^{-1}$

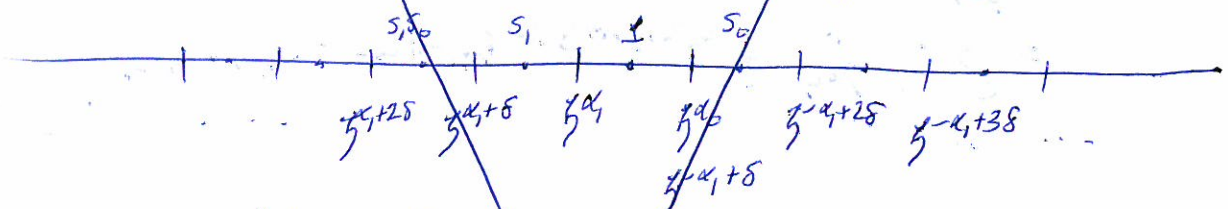
$$1 - x_2 x_1^{-1} - x_3 x_2^{-1} + (x_2 x_1^{-1})^2 (x_3 x_2^{-1}) + (x_3 x_2^{-1})^2 (x_2 x_1^{-1}) - (x_2 x_1^{-1})^2 (x_3 x_2^{-1})^2$$
$$= (1 - x_2 x_1^{-1})(1 - x_3 x_2^{-1})(1 - x_2 x_1^{-1} x_3 x_2^{-1}).$$

and multiply both sides by $x_1^2 x_2$ and this becomes

$$x_1^2 x_2 - x_1 x_2^2 - x_1^2 x_3 + x_2^2 x_3 + x_3^2 x_1 - x_3^2 x_2$$
$$= (x_1 - x_2)(x_2 - x_3)(x_1 - x_3).$$



Example The affine Weylgroup of type $\tilde{A}_1$



$$\sum_{\substack{\text{chambers}\\ w}} (-1)^{\#\text{ of hyp between } 1 \text{ and } w}\, e^{-(\text{hyps between } 1 \text{ and } w)}$$

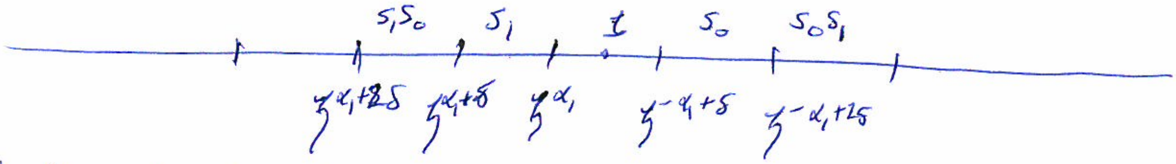
$$= (-1)^0 e^0 + (-1)^1 e^{-\alpha_1} + (-1)^1 e^{-(-\alpha_1+\delta)}$$
$$+ (-1)^2 e^{-(\alpha_1+(\alpha_1+\delta))} + (-1)^2 e^{-(-\alpha_1+\delta-\alpha_1+2\delta)}$$
$$+ (-1)^3 e^{-(\alpha_1+(\alpha_1+\delta)+(\alpha_1+2\delta))} + (-1)^3 e^{-(-\alpha_1+\delta-\alpha_1+2\delta-\alpha_1+3\delta)}$$
$$+ \dots$$

$$= 1 - e^{-\alpha_1} - e^{\alpha_1}e^{-\delta} + e^{-2\alpha_1}e^{-\delta} + e^{-2\alpha_1}e^{-3\delta} - e^{-3\alpha_1}e^{3\delta} - e^{-3\alpha_1}e^{-6\delta} + \dots$$

$$= \prod_{\text{hyperplanes } \zeta^\alpha} (1-e^{-\alpha}) \cdot (\text{Extra factor})$$

$$= \prod_{\ell \in \mathbb{Z}_{\geq 0}} \left(1-e^{-(-\alpha_1+\ell\delta)}\right) \prod_{\ell \in \mathbb{Z}_{\geq 0}} \left(1-e^{-(\alpha_1+\ell\delta)}\right) \cdot \prod_{\ell \in \mathbb{Z}} (1-e^{-\delta})$$

The affine Weyl group of type A_1



has $s_0^2=1,\ s_1^2=1$

$$\sum_{\substack{\text{chambers}\\ w}} (-1)^{(\#\text{ of hyp between } 1 \text{ and } w)} e^{-(\text{hyps between } 1 \text{ and } w)}$$

$$\begin{aligned}
&= (-1)^0 e^0 + (-1)^1 e^{-\alpha_1} + (-1)^1 e^{-(-\alpha_1+\delta)} \\
&\quad + (-1)^2 e^{-2\alpha_1-\delta} + (-1)^2 e^{-(-2\alpha_1+3\delta)} \\
&\quad + (-1)^3 e^{-3\alpha_1-3\delta} + (-1)^3 e^{-(-3\alpha_1+6\delta)} \\
&\quad + \cdots
\end{aligned}$$

Change variable $e^{-\delta}=q$ and $e^{\alpha_1}=z$ to get

$$1 - z^{-1} - zq + z^{-2}q + z^2q^3 - z^{-3}q^3 - z^3q^6 + \cdots$$

$$= \sum_{m\in\mathbb{Z}} (-1)^m z^m q^{\frac{1}{2}m(m-1)}.$$

is equal to

$$\prod_{\substack{\text{hyperplanes}\\ \mathfrak{h}^\alpha}} (1-e^{-\alpha}) \cdot (\text{EXTRA FACTOR})$$

$$= \prod_{\ell\in\mathbb{Z}_{>0}} \left(1-e^{-(-\alpha_1+\ell\delta)}\right) \prod_{\ell\in\mathbb{Z}_{\geq 0}} \left(1-e^{-(\alpha_1+\ell\delta)}\right) \cdot \prod_{\ell\in\mathbb{Z}_{>0}} \left(1-e^{-\ell\delta}\right)$$

$$= \prod_{\ell=1}^{\infty} (1-zq^\ell)(1-z^{-1}q^{(\ell-1)}) \prod_{\ell=1}^{\infty} (1-q^\ell)$$

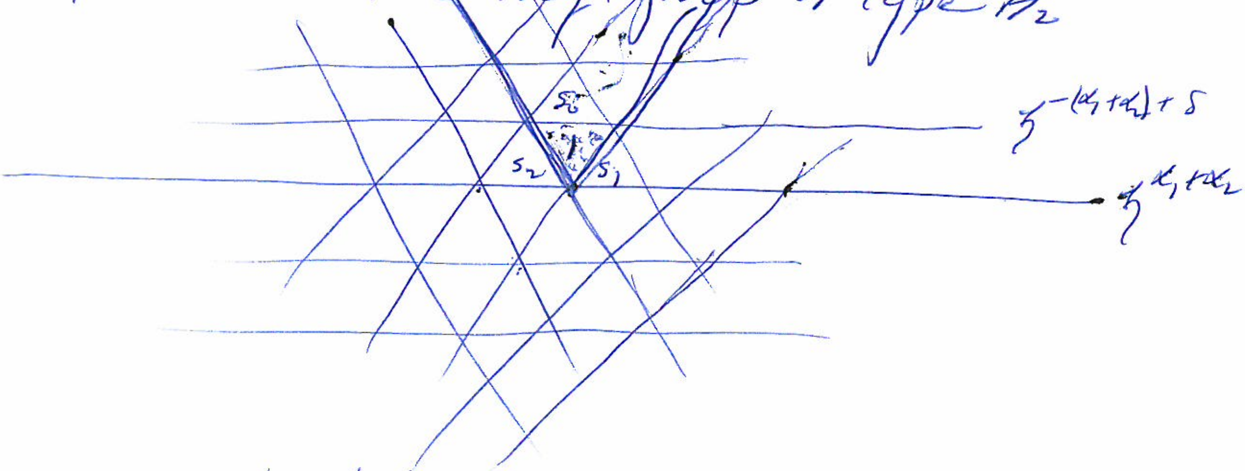
This is the Jacobi-Triple Product identity

$$\prod_{l=1}^{\infty}(1-q^{l})\prod_{l\in\mathbb{Z}_{>0}}(1-q^{l-1}z)(1-q^{l}z^{-1})$$

$$=\sum_{m\in\mathbb{Z}}(-1)^{m}q^{\frac{1}{2}m(m-1)}z^{m}.$$

This is the <u>Jacobi triple product identity</u>

<u>Example</u> The affine Weyl group of type A_2



$$\sum_{\substack{\text{chambers}\\ w}}(-1)^{\#\text{hyps between } 1 \text{ and } w}\, e^{-(\text{hyps between } 1 \text{ and } w)}$$

$$=(-1)^{0}e^{0}+(-1)^{1}e^{-\alpha_1}+(-1)^{1}e^{-\alpha_2}+(-1)^{1}e^{-(-(\alpha_1+\alpha_2)+\delta)}+\cdots$$

$$=1-e^{-\alpha_1}-e^{-\alpha_2}-e^{\alpha_1+\alpha_2}e^{-\delta}+\cdots$$

$$=1-x_2x_1^{-1}-x_3x_2^{-1}-x_1x_3^{-1}q+\cdots$$

$$=\prod_{l\in\mathbb{Z}_{\geq 0}}(1-e^{-\alpha_1+l\delta})(1-e^{-\alpha_2+l\delta})(1-e^{-(\alpha_1+\alpha_2)+l\delta})$$

$$\cdot\prod_{l\in\mathbb{Z}_{>0}}(1-e^{\alpha_1+l\delta})(1-e^{\alpha_2-l\delta})(1-e^{\alpha_1+\alpha_2+l\delta})$$

$$\cdot\prod_{l\in\mathbb{Z}_{>0}}(1-e^{-l\delta})^{2}.$$