Towards elliptic Chevalley groups and flag varieties

Motivating ideas

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$\mathbb{F} = \mathbb{G}_a$	$\mathbb{F}^\times = \mathbb{G}_m$	$E_\tau = \mathbb{C}/_{\mathbb{Z}+\tau\mathbb{Z}}$
additive group	multiplicative group	elliptic curve
$H^*_{S^1}(pt)$ $= \mathbb{C}[x] = \mathcal{O}_{\mathbb{G}_a}$	$K^*_{S^1}(pt)$ $= \mathbb{C}[z,z^{-1}] = \mathcal{O}_{\mathbb{G}_m}$	$Ell_{S^1}(pt)$ $= \mathcal{O}_{E_\tau}$
cohomology	K-theory	Elliptic cohomology

If $G/B = K/T = K/T$ is the flag variety for

G = complex reductive algebraic group

$\cup\!|$

B = Borel subgroup

$\cup\!|$

T = maximal torus

then

$H^*_T(G/B)$	$K_T(G/B)$	$Ell_T(G/B)$

is a module for the

nil Hecke algebra	nil affine Hecke algebra	nil double affine Hecke algebra.

Affine and Elliptic "Chevalley" groups

$$M = SL_3(\mathcal{O}_{E_\tau})$$
$$\cup\!\mid$$
$$L = SL_3(\mathbb{F}((t))) = SL_3(\mathcal{O}_{\mathbb{G}_m})$$
$$\cup\!\mid$$
$$K = SL_3(\mathbb{F}[[t]]) = SL_3(\mathcal{O}_{\mathbb{G}_a}) \xrightarrow[\Phi]{t=0} SL_3(\mathbb{F}) = \bar{K}$$
$$\cup\!\mid \qquad\qquad\qquad\qquad \cup\!\mid$$
$$\mathcal{I} = \Phi^{-1}(\bar{\mathcal{I}}) \longrightarrow \bar{\mathcal{I}} = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ & & * \end{pmatrix}\right\}$$

The Hecke algebra is

$$H_0 = \mathrm{End}_G(\mathbb{C}(G/B)) = \mathrm{End}_G(\mathrm{Ind}_B^G(\mathrm{triv}))$$

for $G = SL_3(\mathbb{F}_q)$.

The affine Hecke algebra is

$$H = \mathrm{End}_L(C(L/\mathcal{I})) = \mathrm{End}_L(C_c(SL_3(\mathbb{Q}_p)/\mathcal{I}))$$

where $L = SL_3(\mathbb{F}_q((t)))$ "=" $SL_3(\mathbb{Q}_p)$.

The double affine Hecke algebra is

$$\tilde{H} = \mathrm{End}_M(C(M/\mathcal{I}))???$$

The flag varieties G/B and L/I

For $c \in \mathbb{F}$ let

$$x_i(c) = \begin{array}{c} \\ i \\ i+1 \\ \\ \end{array} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & c & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \quad \text{and} \quad s_i = \begin{array}{c} \begin{array}{cc} i & i+1 \end{array} \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & & \\ & & & 0 & 1 & \\ & & & -1 & 0 & \\ & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix} \end{array}$$

Then

$$G = \bigsqcup_{w \in W_0} BwB \quad \text{and if } w = s_{i_1} \cdots s_{i_\ell}$$

is reduced then

$$BwB = \{ x_{i_1}(c_1) s_{i_1} \cdots x_{i_\ell}(c_\ell) s_{i_\ell} B \mid c_1, \ldots, c_\ell \in \mathbb{F} \}$$

For $c \in \mathbb{F}$ let

$$x_0(c) = \begin{pmatrix} 1 & & \\ & \ddots & \\ ct & & 1 \end{pmatrix} \quad \text{and} \quad s_0 = \begin{pmatrix} 0 & & & -t^{-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ t & & & & 0 \end{pmatrix}$$

Then

$$L = \bigsqcup_{w \in W} IwI \quad \text{and if } w = s_{i_1} \cdots s_{i_\ell}$$

is reduced then

$$IwI = \{ x_{i_1}(c_1) s_{i_1} \cdots x_{i_\ell}(c_\ell) s_{i_\ell} I \mid c_1, \ldots, c_\ell \in \mathbb{F} \}$$

<u>Point</u> Points in L/I are

labeled paths of type $i_1, \ldots, i_\ell$

for $w = s_{i_1} \cdots s_{i_\ell}$ reduced

MV intersections and Littelmann paths

Let $U^- = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \,\middle|\, * \in \mathbb{F}((t)) \right\} \subseteq L.$

Then

$$L = \bigsqcup_{v \in W} U^- v I = \bigsqcup_{w, v \in W} (IwI \cap U^- v I)$$

and there is an explicit folding procedure

labeled path of type $i_1, \ldots, i_\ell$ $\longrightarrow$ folded labeled path of type $i_1, \ldots, i_\ell$

so that

$$IwI \cap U^- v I = \left\{ x_{i_1}(c_1) s_{i_1} \cdots x_{i_\ell}(c_\ell) s_{i_\ell} I \,\middle|\, \begin{array}{c} \text{folding of} \\ p \text{ ends in} \\ v \end{array} \right\}$$

The image of the "forget the labels" map

$$\left\{ \begin{array}{c} \text{folded labeled} \\ \text{paths} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{folded} \\ \text{paths} \end{array} \right\}$$

is the set of Littelmann paths.

Transition matrices

The affine Hecke algebra

$$H = C(I\backslash L/I)$$

acts on

$C(I\backslash L/I) = \mathrm{span}\{T_w \mid w\in W\}$ with $T_w = \chi_{IwI}$

and on

$C(U^-\backslash L/I) = \mathrm{span}\{X_v \mid v\in W\}$ with $X_v = \chi_{U^-vI}$

and if $w = s_{i_1}\cdots s_{i_\ell}$ is reduced

$$T_w = \sum_{\substack{\text{folded paths of}\\ \text{type } i_1,\dots,i_\ell \text{ with}\\ \text{positive folds}}} \mathrm{Card}\left\{\begin{matrix}\text{possible labelings}\\ \text{of } p\end{matrix}\right\} X_{\mathrm{end}(v)}$$

The double affine Hecke algebra $\tilde{H} \overset{?}{=} C(I\backslash M/I)$

acts on the polynomial representation $\tilde{H}\mathbb{1}$.

$\tilde{H}\mathbb{1}$ has two bases

$$\{E_\lambda\} \quad \text{and} \quad \{X_\lambda\}$$

and

$$E_\lambda = \sum_{\substack{\text{folded paths of}\\ \text{type } i_1,\dots,i_\ell \text{ with}\\ \text{positive and negative folds}}} \mathrm{wt}(p)\, X_{\mathrm{end}(p)}.$$

The Weyl character formula

$$G/B = SL_3(\mathbb{C})/B = \text{flag variety}$$

There are morphisms of rings

$$\mathrm{Rep}(T) = K_T(pt) \longrightarrow K_T(G/B) \xrightarrow{\pi_!} K_T(G/B)$$

$$\mathbb{C}_\lambda \longmapsto [\mathcal{L}_\lambda] = [G \times_B \mathbb{C}_\lambda] \longmapsto \sum_{w \in W_0} w\left(\frac{e^{\lambda+\rho}}{a_\rho}\right)$$

where

a_ρ = Weyl denominator, and

$$s_\lambda = \sum_{w \in W_0} w\left(\frac{e^{\lambda+\rho}}{a_\rho}\right) = \text{Weyl character}$$

$$= \sum_{\substack{\text{folded paths} \\ \text{of type } i_1,\ldots,i_\ell \\ \text{of maximal dim}}} e^{\mathrm{end}(p)} \qquad \text{(from Littelmann)}$$

With Nora Ganter we explain:

There are morphisms of sheaves

$$Ell_T(pt) \longrightarrow Ell_T(G/B) \xrightarrow{\pi_!} Ell_T(pt)$$

Theta functions

Restrict to global sections

$$\Gamma(Ell_T(pt)) \longrightarrow \Gamma(Ell_T(G/B)) \longrightarrow \Gamma(Ell_T(pt))$$

$$\Theta_{\lambda,k} \longmapsto \sum_{w \in W_0} w\left(\frac{\Theta_{\lambda+\rho,\, k+g}}{\Theta_{\rho,g}}\right)$$

to see the picture of

Looijenga (Roots systems and Elliptic curves)

and

Kac-Peterson (Affine Lie algebras and modular forms)

as a consequence of an appropriate

Atiyah-Segal-Lefschetz fixed pt formula

for elliptic cohomology.