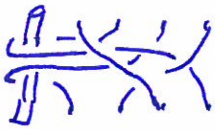
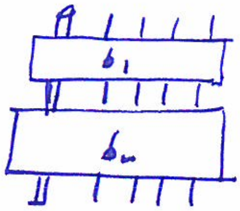
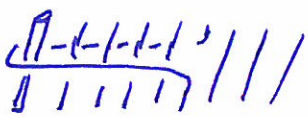
What are KLR BMW algebras?

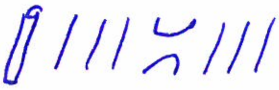
International Conference on Non-Commutative rings and Combinatorial Representation Theory

The affine braid group $\mathcal{B}_k$

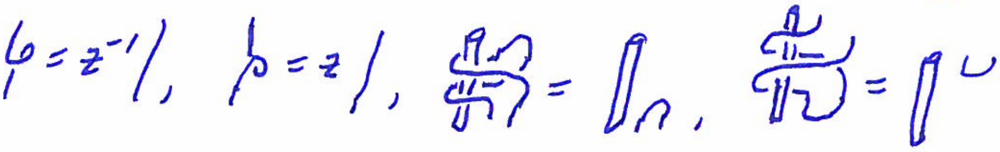
$b =$  $\in \mathcal{B}_4$ and $b_1 b_2 =$ 

Fix constants q and z. Let

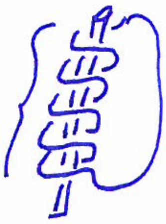
$T_i =$  and $Y_i = z$ 

$E_i =$  given by $T_i Y_i = Y_{i+1} T_i - (q - q^{-1})(1 - E_i)$.

The affine BMW algebra $\mathcal{W}_k$ is $\mathbb{C}\mathcal{B}_k$ with



and

l twists  $= z_1^{(l)}$  where $z_1^{(l)}$ are constants

The affine Hecke algebra $\mathcal{H}_k$ is $\mathcal{W}_k$ with

$E_i = 0$.

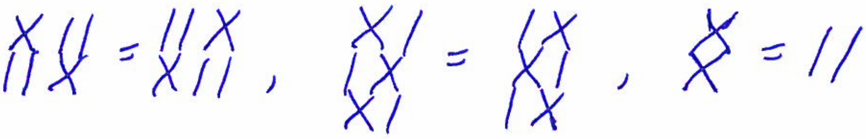
For some choices of $z_1^{(l)}$, $\mathcal{W}_k = \mathcal{H}_k$.

The <u>degenerate</u> <u>affine</u> <u>braid</u> <u>group</u> B_k

is the algebra generated by

$t_{s_i} =$  and $y_i =$ 

with relations



$y_iy_j = y_jy_i$ and $t_{s_i}(y_i+y_{i+1}) = (y_i+y_{i+1})t_{s_i}$

if $t_{i\,i+1} = y_{i+1} - t_{s_i}y_it_{s_i}$ then $t_{s_i}t_{s_{i+1}}(t_{i\,i+1})t_{s_{i+1}}t_{s_i} = t_{i+1,i+2}$

Fix $\epsilon = \pm 1$ and define $e_i =$  by

$$t_{s_i}y_i = y_{i+1}t_{s_i} - (1-e_i)$$

The <u>degenerate</u> <u>affine</u> <u>BMW</u> <u>algebra</u> is B_k with

$t_{s_i}e_i = \epsilon e_i = e_it_{s_i}$, $e_it_{s_{i\pm1}}e_i = \epsilon e_i$

$e_i(y_i+y_{i+1}) = (y_i+y_{i+1})e_i = 0$ and

$e_1y_1^{\ell}e_1 = z_1^{(\ell)}e_1$ where $z_1^{(\ell)}$ are constants.

The <u>degenerate</u> <u>affine</u> <u>Hecke</u> <u>algebra</u> H_k is W_k

with $e_i = 0$.

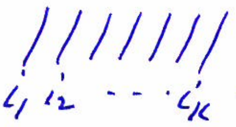
KLR algebras

Let $I = \mathbb{Z}/e\mathbb{Z}$, $e \in \mathbb{Z}_{\geq 0}$, $e \neq 2$ ($e=0$ is the most. imp. case).

Fix symbolse α_i, $i \in I$, and

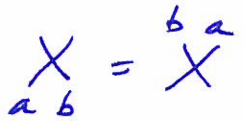
$$\alpha = \sum_{i \in I} m_i \alpha_i, \text{ with } m_i \in \mathbb{Z}_{\geq 0} \text{ and } \sum_{i \in I} m_i = k.$$

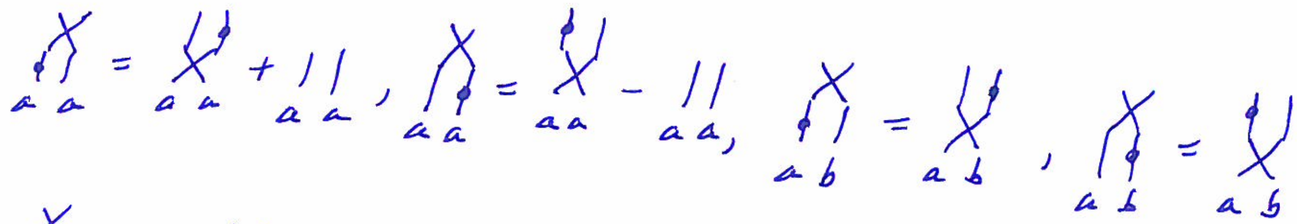
The KLR algebra R_α has generators

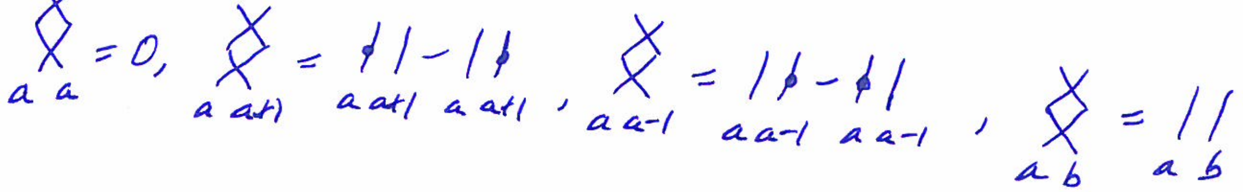
$e_{i_1 \cdots i_k} =$ , $x_i =$ , $\psi_l =$ 

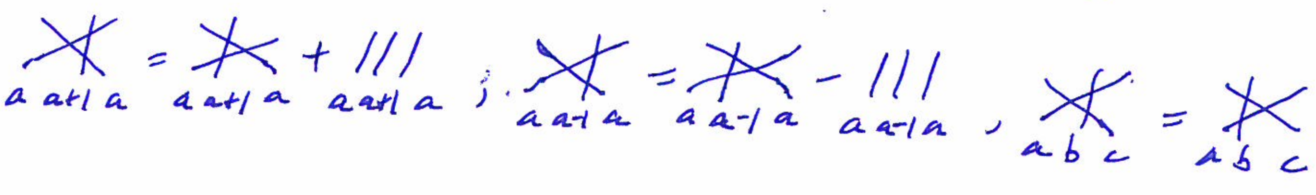
for $\alpha_{i_1} + \cdots + \alpha_{i_k} = \alpha$, with relations

$$e_u e_v = \delta_{uv} e_u, \quad \sum_{\substack{u = i_1 \cdots i_k \\ \alpha_{i_1} + \cdots + \alpha_{i_k} = \alpha}} e_u = 1, \quad x_i x_j = x_j x_i, \quad x_i e_u = e_u x_i$$

$\psi_l e_u = e_{s_l u} \psi_l$, i.e. 







MOST IMPORTANT

(1) R_α is a $\mathbb{Z}$-graded algebra

$$\deg\Big(\underset{i_1\, i_2\, \cdots\, i_k}{||||||}\Big) = 0, \quad \deg\big(||\bullet|\bullet|||\big) = 2$$

$$\deg\Big(\underset{a\ a}{X}\Big) = -2, \quad \deg\Big(\underset{a\ a\pm 1}{X}\Big) = 1, \quad \deg\Big(\underset{a\ b}{X}\Big) = 0.$$

(2) There is a character map.

Let F be the free associative algebra generated by f_i, $i \in I$.

Let M be a $\mathbb{Z}$-graded R_α-module $M = \bigoplus_{\ell \in \mathbb{Z}} M[\ell]$

$$\mathrm{char}(M) = \sum_{\ell \in \mathbb{Z}} \sum_{\substack{i_1 \cdots i_k \\ \alpha_{i_1} + \cdots + \alpha_{i_k} = \alpha}} \dim(e_{i_1 \cdots i_k} M[\ell])\, q^\ell f_{i_1} \cdots f_{i_k}$$

so that $\mathrm{char} : \left\{ \begin{matrix} \mathbb{Z}\text{-graded} \\ R_\alpha\text{-modules} \end{matrix} \right\} \longrightarrow F$

The multiplication on $\{\mathbb{Z}\text{graded } R_\alpha\text{-modules}\}$ is

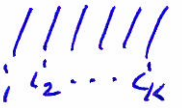
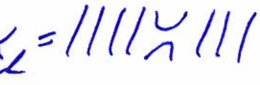
$$M \circ N = \mathrm{Ind}_{R_\alpha \otimes R_\beta}^{R_{\alpha+\beta}} (M \otimes N)$$

and the induced product on F is the <u>q-shuffle product</u>

The image of char is the <u>quantum group</u>.

What are KLR BMW algebras?

P_α is generated by

$p_{i_1 \cdots i_k} =$ , $x_i =$ , $\psi_\ell =$ , $\kappa_\ell =$ 

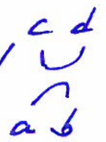
(2) is obtained by the Key lemma

$R_{\alpha+\beta}$ is a free $R_\alpha \otimes R_\beta$ module with basis

$\{ \psi_w \mid w \in S_{k+\ell}/S_k \times S_\ell \}$ where

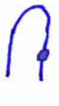
$\psi_w = \psi_{\ell_1} \cdots \psi_{\ell_r}$ for a fixed reduced word $w = s_{\ell_1} \cdots s_{\ell_r}$

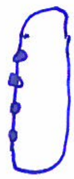
We need

(1) $\deg\Big($  $\Big) = ???$

and relations

 $= 0$ unless $b = r - a$,

 $\overset{?}{=} -$ ,  $= ???$ $\cap$,  $= ???$ $|$

 $= ???$

We want

$\bigoplus_{\substack{\alpha \\ ht(\alpha)=k}} P_\alpha$ "$\simeq$" W_k where "$\simeq$" means 'have the same finite dimensional representations'.

where $ht(\alpha) = k$ with $\alpha = \sum_{i \in I} m_i \alpha_i$ with $\sum_{i \in I} m_i = k$.