Elliptic cohomology and Weyl character formulas

Motivation

Univ. of Adelaide, Workshop on Dirac operators in Geometry, Topology, Representation Theory and Physics

$\mathbb{F} = \mathbb{G}_a$ additive ~~multiplicative~~ group

$\mathbb{F}^\times = \mathbb{G}_m$ multiplicative group

$E_\tau = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ elliptic curve.

$H_{S^1}(pt) = \mathcal{O}_{\mathbb{G}_a} = \mathbb{C}[x]$ cohomology

$K_{S^1}(pt) = \mathcal{O}_{\mathbb{G}_m} = \mathbb{C}[z, z^{-1}]$ K-theory

$E_{S^1}(pt) = \mathcal{O}_{E_\tau}$. Elliptic cohomology.

Chevalley groups

$$M = SL_3(\mathcal{O}_E)$$
$$\cup\!|$$
$$L = SL_3(\mathbb{F}((t))) = SL_3(\mathcal{O}_{\mathbb{G}_m})$$
$$\cup\!|$$
$$K = SL_3(\mathbb{F}[[t]]) = SL_3(\mathcal{O}_{\mathbb{G}_a}) \xrightarrow[\Phi]{t=0} SL_3(\mathbb{F}) = \bar{K} = G$$
$$\cup\!| \qquad\qquad\qquad \cup\!|$$
$$I = \Phi^{-1}(\bar{I}) \longrightarrow \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\} = \bar{I} = B$$

$G/B = \bar{K}/\bar{I} = K/I$ is the flag variety

L/K = loop Grassmannian, L/I = affine flag variety.

M/L = elliptic Grassmannian, M/K = ??, M/I = elliptic flag variety.

Hecke algebras

The finite Hecke algebra is

$$H_0 = End_G(C(G/B)) = End_G(Ind_B^G(triv))$$

The affine Hecke algebra is

$$H = End_L(C(L/I)) = End_L(C_c(SL_3(\mathbb{Q}_p)/I))$$

where $L = SL_3(\mathbb{F}_p((t)))$ "=" $SL_3(\mathbb{Q}_p)$.

The double affine Hecke algebra of Cherednik

is $\tilde{H} \overset{?}{=} End_H(C(M/I))$.

The Bruhat decomposition is

$$G = \bigsqcup_{w \in W_0} BwB, \qquad W_0 \text{ has generators } s_1, \ldots, s_n$$

with relations $s_i^2 = 1$ and $\underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}$.

and H_0 has generators $T_1, \ldots, T_n$

and relations

$$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1, \quad \underbrace{T_i T_j \cdots}_{m_{ij}} = \underbrace{T_j T_i \cdots}_{m_{ij}}.$$

W_0 acts on $\mathfrak{h}_{\mathbb{Z}}^*$.

~~The~~ affine Hecke algebra

$$\mathbb{C}[X] = \mathrm{span}\{X^\lambda \mid \lambda \in \mathfrak{h}_\mathbb{Z}^*\} \text{ with } X^\lambda X^\mu = X^{\lambda+\mu}$$

Then

H is generated by subalgebras H_0 and $\mathbb{C}[X]$

with

$$T_i X^\lambda = X^{s_i\lambda} T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X^{-\alpha_i}} (X^\lambda - X^{s_i\lambda})$$

and

$\{X^\mu T_w \mid \mu \in \mathfrak{h}_\mathbb{Z}^*, w \in W_0\}$ is a basis of H

The double affine Hecke algebra $\tilde{H}$ has basis

$$\{q^\ell X^\mu T_w Y^{\lambda^\vee} \mid \ell \in \mathbb{Z}, \mu \in \mathfrak{h}_\mathbb{Z}^*, \lambda^\vee \in \mathfrak{h}_\mathbb{Z}, w \in W_0\}$$

with $\quad X^\mu Y^{\lambda^\vee} = q^{\langle \mu, \lambda^\vee \rangle} Y^{\lambda^\vee} X^\mu +$ "lower terms".

Let

$$u_0 = \sum_{w \in W_0} w \quad \text{and} \quad e_0 = \sum_{w \in W_0} \det(w)\, w \quad \text{in } \mathbb{C}W_0$$

so that $\quad w u_0 = u_0 \quad$ and $\quad e_0 w = (-1)^{\ell(w)} w$.

Let

$$\mathbb{1}_0 = \sum_{w \in W_0} (t^{\frac{1}{2}})^{\ell(w)} T_w \quad \text{and} \quad \varepsilon_0 = \sum_{w \in W_0} (-t^{\frac{1}{2}})^{\ell(w)} T_w$$

so that

$$T_w \mathbb{1}_0 = (t^{\frac{1}{2}})^{\ell(w)} \mathbb{1}_0 \quad \text{and} \quad \varepsilon_0 T_w = (-t^{-\frac{1}{2}})^{\ell(w)} \varepsilon_0$$

Hecke algebras and Geometric Langlands

$$\mathbb{C}[X]^{W_0} = Z(H) \longrightarrow \mathbb{1}_0 H \mathbb{1}_0 \longrightarrow \varepsilon_0 H \mathbb{1}_0$$

$$P_\lambda(0,t) \longleftarrow \mathbb{1}_0 X^\lambda \mathbb{1}_0$$

$$s_\lambda \longleftarrow C_\lambda \longleftarrow A_{\lambda+\rho} = \varepsilon_0 X^{\lambda+\rho} \mathbb{1}_0$$

$$f \longmapsto f \mathbb{1}_0 \longmapsto A_\rho f \mathbb{1}_0$$

$$A_\rho = \prod_{\alpha \in R^+} (t^{\frac{1}{2}} X^{\alpha/2} - t^{-\frac{1}{2}} X^{-\alpha/2}) = (t^{\frac{1}{2}})^{\ell(w_0)} X^\rho \prod_{\alpha \in R^+} (1 - t^{-1} X^{-\alpha})$$

$P_\lambda(0,t)$ is Macdonald's spherical function.

$H\mathbb{1}_0$ is the polynomial representation of H

C_λ is the Kazhdan-Lusztig basis element for the spherical Hecke algebra $\mathbb{1}_0 H \mathbb{1}_0$

$$\mathbb{1}_0 H \mathbb{1}_0 = \mathrm{End}_L(C_c(L/K)) = C(K \backslash L / K)$$

Hermann-Weyl

$$u_0 \mathbb{C}[X] = \mathbb{C}[X]^{W_0} \longrightarrow \mathbb{C}[X]^{\det} = e_0 \mathbb{C}[X]$$

$$f \longmapsto a_\rho f$$

$$s_\lambda \longleftarrow a_{\lambda+\rho} = e_0 X^{\lambda+\rho}$$

$$u_0 X^\lambda = m_\lambda$$

$$a_\rho = \prod_{\alpha \in R^+} (X^{\alpha/2} - X^{-\alpha/2}) = X^\rho \prod_{\alpha \in R^+} (1 - X^{-\alpha})$$

(see arXiv 0401298 with Nelsen)

Theta functions

$\mathbb{C}[X]$ has basis $\{X^\mu \mid \mu \in \bar{\mathfrak{h}}_{\mathbb{Z}}^*\}$

$\mathbb{C}[X]^{W_0}$ has basis $\{m_\mu \mid \mu \in \bar{\mathfrak{h}}_{\mathbb{Z}}^*/W_0\}$

$\mathbb{C}[X]^{\det}$ has basis $\{a_{\mu+\rho} \mid \mu \in \bar{\mathfrak{h}}_{\mathbb{Z}}^*/W_0\}$

Kac-Peterson, Mumford, Looijenga tell us

$$A_\tau = \bar{\mathfrak{h}}_{\mathbb{C}}^* \Big/ \mathfrak{h}_{\mathbb{Z}} \oplus \tau \mathfrak{h}_{\mathbb{Z}} = E_\tau \otimes \mathfrak{h}_{\mathbb{Z}}$$ are a family of abelian varieties

Th_m = Theta functions of degree m $= H^0(A_\tau, \mathcal{L}^m)$

has basis $\{\Theta_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^* \bmod \bar{\mathfrak{h}}_{\mathbb{Z}}^* \bmod \mathbb{C}\delta \mid \lambda(c) = m\}$

$Th_m^{W_0}$ has basis $\{M_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*/W \bmod \mathbb{C}\delta,\ \lambda(c) = m\}$

$Th_m^{\det}$ has basis $\{A_{\lambda+\hat\rho} \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*/W \bmod \mathbb{C}\delta,\ \lambda(c) = m\}$

$$Th^{W_0} = \bigoplus_{w \in W_0} Th_m^{W_0} \xrightarrow{\sim} Th^{\det} = \bigoplus_m Th_m^{\det}$$

$$\sum_{w \in W_0} \Theta_{w\lambda} = M_\lambda$$

Weyl character for $(LG)^\tau$ $\quad s_\lambda \longleftarrow A_{\lambda+\hat\rho} = \sum_{w \in W_0} \Theta_{w\lambda} \det(w)$

$$f \longmapsto A_{\hat\rho} f$$

$A_{\hat\rho}$: Weyl denominator for $(LG)^\tau$

and $A_{\hat\rho}$ has divisor corresponding to

$\mathcal{L}^g$ where g = dual Coxeter number.

Pushforward $\quad \pi : G/B \longrightarrow pt$

$$K_T(pt) \xrightarrow{\pi^*} K_T(G/B) \xrightarrow{\pi_!} K_T(pt)$$

$$\mathbb{C}_\lambda \longmapsto [G \times_B \mathbb{C}_\lambda] \longmapsto s_\lambda = \text{Weyl character.}$$

Better to write this as

$$\begin{tikzcd}
K_T(pt) = K_G(G/T) \arrow[r, "\sim"] \arrow[d, "\mathrm{res}"'] & \hat{K}_G((G/T)^{-\tau}) \arrow[d, "\mathrm{res}"] & \\
K_T(G/T) \arrow[r, "\sim"] & \hat{K}_T((G/T)^{-\tau}) \arrow[r] \arrow[d] & K_T(pt) \\
& \hat{K}_T(((G/T)^T)^{-\tau}) \arrow[r] & \hat{K}_T((G/T)^T) \arrow[u]
\end{tikzcd}$$

and, in elliptic cohomology

$$\begin{tikzcd}
E_T(pt) = E_G(G/T) \arrow[r, "\sim"] \arrow[d] & \hat{E}_G((G/T)^{-\tau}) \arrow[d, "\mathrm{res}"] & \\
E_T(G/T) \arrow[r, "\sim"] & \hat{E}_T((G/T)^{-\tau}) \arrow[r] \arrow[d] & K_T(pt) \\
& \hat{E}_T(((G/T)^T)^{-\tau}) \arrow[r] & E_T((G/T)^T) \arrow[u]
\end{tikzcd}$$

and

$$\mathcal{L}^m \otimes E_T(pt) = \mathcal{O}_{A_T} \otimes \mathcal{L}^m = \mathcal{L}^m .$$

$$\mathcal{L}^m \otimes \hat{E}_G((G/B)^{-\tau}) = \mathcal{L}^g \otimes \mathcal{L}^m = \mathcal{L}^{g+m},$$

IGA/AMSI Workshop

Dirac Operators in Geometry, Topology, Representation Theory, and Physics

18-22 October 2010

Program Schedule

Monday, 18 October 2010

Time	Speaker	Title
09:30	*Refreshments*	
09:50	*Opening*	
10:00	Dan Freed	Introduction to twisted K-theory
12:00	*Lunch Break*	
14:00	Frédéric Rochon	Dirac operators on manifolds with foliated boundaries
15:00	*Refreshments*	
15:30	David Ridout	D-brane charges in Wess-Zumino-Witten models
16:30	*Reception*	

Tuesday, 19 October 2010

Time	Speaker	Title
09:30	*Refreshments*	
10:00	Dan Freed	Loop groups and Dirac families
12:00	*Lunch Break*	
14:00	Arun Ram	Elliptic cohomology and Weyl character formulas
15:00	*Refreshments*	
15:30	Anne Thomas	Lattices in complete Kac-Moody groups
18:00	*Workshop dinner*	