A probabilistic interpretation of Macdonald polynomials (1)

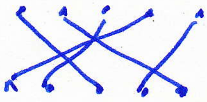
Markov chains

jt. with Persi Diaconis

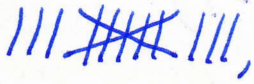
Combinatorial Representation Theory Day

Hannover 18 Feb. 2011.

State space: $\{w \mid w \in S_n\}$

$w =$  $\in S_5$, the symmetric group

Operator: $M = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} s_{ij}$,

where $s_{ij} =$ , the transposition switching i + j.

Starting state: 1

The story: A deck of cards, choose 2, cards i and j and switch them.

How long does it take the deck to get random?

The stationary distribution:

$$\pi = \frac{1}{n!} \sum_{w \in S_n} w, \quad \text{the uniform distribution.}$$

Distances to stationarity

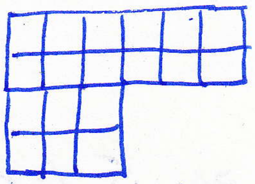
$$4\|M^\ell 1 - \pi\|_{TV}^2 = \Big(\sum_{y \in S_n} |M^\ell(1,y) - \pi(y)|\Big)^2 \qquad L^1\text{-norm}$$

$\le$

$$\|M^\ell 1 - \pi\|_2^2 = \sum_{y \in S_n} \frac{(M^\ell 1(y) - \pi(y))^2}{\pi(y)} \qquad L^2\text{-norm.}$$

Lumping

New state space: $\{p_\mu \mid \mu \text{ is a partition of } n\}$

$\mu =$  $= (6,6,3,3) = 1^0 2^0 3^2 4^0 5^0 6^2$ has $n=18$

$$p_\mu = \frac{z_\lambda}{n!} \sum_{\tau(w)=\mu} w, \quad \text{where } \tau(w) = \text{cycle type of } w.$$

form a basis of the center of the group algebra $\mathbb{C}S_n$

New chain: Same as the old but only reports $\tau(w)$

$$M \cdot x = \sum_{y} M(x,y)\, y,$$ $M(x,y)$ is the probability of moving from x to y.

Eigenvectors and eigenvalues

$$s_\lambda = \sum_\mu \chi^\lambda_\mu p_\mu \quad \text{and} \quad M s_\lambda = \chi^\lambda(s_{12}) s_\lambda$$

where $\chi^\lambda_\mu = \mathrm{Tr}(w, S_n^\lambda)$ are the characters of the irreducible S_n-modules S_n^λ.

$$\chi^\lambda(s_{12}) = \left(\begin{array}{c}\text{sum of the}\\ \text{contents of the}\\ \text{boxes in } \lambda\end{array}\right)$$

Convergence of M^ℓ is controlled by the second largest eigenvalue.

The Metropolis algorithm (following Hanlon ...)

Fix α, $0<\alpha<1$. A step of M_α is:

Given a deck at state w, choose i and j.

- if $\tau(s_{ij}w) = \tau(w)+1$, move to $s_{ij}w$
- if $\tau(s_{ij}w) = \tau(w)-1$, move to $s_{ij}w$ with probability $1/\alpha$.

The new chain M_α has:

stationary distribution: $\pi_\alpha = \frac{1}{\text{const}} \sum_{\lambda \in \mathcal{P}_n} \alpha^{-\ell(\lambda)} z_\lambda p_\lambda$

eigenvectors: $J_\lambda^\alpha = \sum_{\mu \in \mathcal{P}_n} \chi_\mu^\lambda(\alpha) p_\mu$, Jack polynomials

eigenvalues: $M_\alpha J_\lambda^\alpha = \beta_\lambda(\alpha) J_\lambda^\alpha$

where $\ell(\lambda) = \#$ of parts of λ, and

$$\beta_\lambda(\alpha) = \sum_{i=1}^{n} \alpha\lambda_i + n - i \quad ????$$

Unknowing to polynomials

In symmetric function theory

$$p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_\ell}, \quad \text{for } \mu = (\mu_1, \dots, \mu_\ell)$$

where

$$p_k = x_1^k + \cdots + x_n^k, \quad \text{for } k \in \mathbb{Z}_{>0}.$$

For $\alpha = \frac{1}{2}, 1, 2$ the Jack polynomials J_λ^α are classical spherical functions (zonal polynomials) for

$$GL_n(\mathbb{H})/U_n(\mathbb{H}), \qquad GL_n(\mathbb{C})/U_n(\mathbb{C}), \qquad GL_n(\mathbb{R})/O_n(\mathbb{R})$$

The operator

$$D_\alpha = \frac{\alpha}{2} \sum_{i=1}^{n} x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i}$$

acts on $\mathbb{C}[x_1, \dots, x_n]$ with

eigenvectors J_λ^α and eigenvalues $\beta_\lambda(\alpha)$.

Now we are in the world of

Harmonic analysis: Spectra of Laplacians

Mathematical Physics: Spectra of Hamiltonians.

5

Auxiliary variables = data augmentation
= hit and run.

Defined by Edwards and Sokal (for fast Ising and Potts)

Generalises Swedsen-Wang.

The data:

State space: X Auxiliary sets I

Probability distribution on X: $\pi(x)$

Probability distribution on I: $w_x(i)$
for each $x \in X$

Markov matrix on X: $M_i(x,y)$
for each $i \in I$

such that

$$\pi(x) w_x(i) M_i(x,y) = \pi(y) w_y(i) M_i(y,x)$$

This data defines a Markov chain

$$M(x,y) = \sum_i w_x(i) M_i(x,y)$$

Our Auxiliary variables chain

$$X = \mathcal{P}_n \quad \text{and} \quad \mathcal{I} = \bigcup_{i=1}^{n} \mathcal{P}_i .$$

$$\pi(\lambda) = \frac{\text{const}}{z_\lambda \prod_i \left(\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}\right)}$$

$$w_\lambda(\mu) = \frac{1}{(q^n-1)} \prod_{i=1}^{n} \binom{a_i(\lambda)}{a_i(\mu)} \frac{(q^i-1)^{a_i(\lambda)}}{(q^i-1)^{a_i(\mu)}}, \quad \text{and}$$

$$M_\rho(\lambda,\mu) = \begin{cases} \frac{1}{z_\mu(1-t^{-1})} \prod_i (1-t^{-i})^{a_i(\nu)}, & \text{if } \mu = \rho \cup \nu \\ 0, & \text{otherwise} \end{cases}$$

The story: Start with λ

- Delete some parts to get $\lambda - \gamma$, with probability $w_\lambda(\lambda-\gamma)$
- Add some parts to get μ, with probability $M_{\lambda-\gamma}(\lambda,\mu)$

This gives a Markov chain

$$M_{q,t}(\lambda,\mu) \text{ on } \mathcal{P}_n = \{p_\lambda \mid \lambda \text{ is a partition of } n\}$$

Macdonald polynomials

Theorem The eigenvectors of $M_{q,t}$ are

$$P_\lambda(q,t) = \sum_\mu \chi^\lambda_\mu(q,t)\, p_\mu,$$

the Macdonald polynomials, and

$$M_{q,t} P_\lambda(q,t) = \beta_\lambda(q,t) P_\lambda(q,t)$$

where $\beta_\lambda(q,t) = \sum_{i=1}^{\ell(\lambda)} q^{\lambda_i} t^{n-i}$

Remarks

- $P_\lambda(0,0) = s_\lambda$ = Schur functions
  = characters of compact Lie groups.
- $P_\lambda(0,t)$ = Hall-Littlewood polynomials
  = spherical functions for $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$.
- $\lim_{t\to 1} P_\lambda(t^\alpha, t) = J^\alpha_\lambda$, the Jack polynomials
- For type $(C_n^\vee, C_n)$, $P_\lambda(q,t)$ are the Koornwinder polys.
- For type $(C_1^\vee, C_1)$, $P_\lambda(q,t)$ are the Askey-Wilson polys.

Combinatorial Representation Theory Day

Friday, February 18, 2011

Leibniz Universität Hannover

Institut für Algebra, Zahlentheorie und Diskrete Mathematik

Lectures take place in room **f435 (Stahlbausaal)**, refreshments are provided in room **a410**.

from 10:30	**Welcome coffee**
11:15	Arun Ram (Melbourne, currently Bonn) *A probabilistic interpretation of Macdonald polynomials*
12:20	Christian Gutschwager (Hannover) *Generalised stretched Littlewood-Richardson coefficients*
	Lunch and discussions (in a410)
14:15	Raquel Simoes (Leeds) *Hom configurations and non-crossing partitions*
	Coffee
15:30	Alexander Kleshchev (Eugene/Oregon, currently Bonn) *Group algebras of the symmetric groups and related Hecke algebras as graded algebras*
