Elliptic Schubert Calculus, Peter O'Sullivan conference

Australian National Univ. 2-4 Sept. 2011.

Weyl Character formula

G complex reductive alg. group $\qquad GL_n(\mathbb{C})$

$\cup\!|$ $\qquad \cup\!|$

B Borel subgroup $\qquad \left\{\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}\right\}$

$\cup\!|$ $\qquad \cup\!|$

T maximal torus $\qquad \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$

$$\mathring{\mathfrak{h}}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^\times, T) \quad \text{and} \quad \mathring{\mathfrak{h}}^*_{\mathbb{Z}} = \mathrm{Hom}(T, \mathbb{C}^\times)$$

and

$$W_0 = N(T)/T \text{ is the Weyl group.}$$

$Rep(T) = K_T(pt) = \mathrm{span}\{e^\lambda \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}\}$ with $e^\lambda e^\mu = e^{\lambda+\mu}$

W_0 acts on $Rep(T)$ by $we^\lambda = e^{w\lambda}$ and

$$Rep(G) = K_G(pt) = K_T(pt)^{W_0} = \{f \in K_T(pt) \mid wf = f \text{ for } w \in W_0\}.$$

$K_T(pt)^{\det}$ is a free module of rank 1 over $K_T(pt)^{W_0}$

$$\begin{array}{ccc} K_T(pt)^{W_0} & \xrightarrow{\sim} & K_T(pt)^{\det} \\ f & \longmapsto & a_\rho f \\ s_\lambda & \longleftarrow & a_{\lambda+\rho} \end{array} \quad \text{"naive basis"}, \quad \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}/W_0$$

where $K_T(pt)^{\det} = \{f \in K_T(pt) \mid wf = \det(w) f, \ w \in W_0\}$

$$a_\mu = \sum_{w \in W_0} \det(w) e^{w\mu} \quad \text{and} \quad s_\lambda = [L(\lambda)] \text{ in } K_T(pt),$$

where $L(\lambda)$ are the simple G-modules.

Cohomologies H_T, K_T and Ell_T.

To build Ell, fix an elliptic curve

$$E = \frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}} \quad \text{with} \quad \tau \in \mathcal{H}^+ \quad \text{the upper half plane.}$$

(a) T rank 1 ($T = \mathbb{C}^\times$). Then $\mathfrak{h}_{\mathbb{Z}}^{*} = \mathbb{Z}$ and

$$H_T(pt) = \mathbb{C}[x] \quad \text{and} \quad K_T(pt) = \mathbb{C}[X^{\pm 1}]$$

$$Ell_T(pt) = \mathcal{O}_E$$

(b) T rank ℓ ($T \cong (\mathbb{C}^\times)^\ell$) and $\mathfrak{h}_{\mathbb{Z}}^{*} \cong \mathbb{Z}^\ell = \mathbb{Z}\text{span}\{\omega_1, \ldots, \omega_\ell\}$.
If $X_i^{\pm 1} = e^{\pm \omega_i}$ then

$$H_T(pt) = \mathbb{C}[x_1, \ldots, x_\ell] \quad \text{and} \quad K_T(pt) = \mathbb{C}[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}]$$

$$Ell_T(pt) = \mathcal{O}_{E^\ell} \quad \text{with} \quad E^\ell = \mathfrak{h}_{\mathbb{Z}}^{*} \otimes_{\mathbb{Z}} E = \frac{\mathfrak{h}_{\mathbb{C}}}{\mathfrak{h}_{\mathbb{Z}} + \tau\mathfrak{h}_{\mathbb{Z}}}$$

(c)

$$H_G(pt) = H_T(pt)^{W_0} \quad \text{and} \quad K_G(pt) = K_T(pt)^{W_0}$$

$$Ell_G(pt) = \mathcal{O}_{E^\ell/W_0}$$

(d) $H_T(G/B) = H_T(pt) \otimes_{H_G(pt)} H_T(pt)$ (Borel)

$K_T(G/B) = K_T(pt) \otimes_{K_G(pt)} K_T(pt)$ (Demazure)

$Ell_T(G/B) \overset{?}{=} \mathcal{O}_{E^\ell} \otimes_{\mathcal{O}_{E^\ell/W_0}} \mathcal{O}_{E^\ell}$

Replacing $\mathcal{O}_{E^\ell}$ with Th (Mumford/Igusa/Kac-Peterson)

$\mathring{\mathfrak{h}}_\mathbb{Z}$ comes with a pos. def. symm. bilinear form

$(\,|\,): \mathring{\mathfrak{h}}_\mathbb{Z}\times\mathring{\mathfrak{h}}_\mathbb{Z}\to\mathbb{Z}$, i.e. an

ample line bundle $\mathcal{L}_\tau$ on $\mathring{\mathfrak{h}}_\mathbb{C}/\mathring{\mathfrak{h}}_\mathbb{Z}+\tau\mathring{\mathfrak{h}}_\mathbb{Z}$.

$$Th(\tau)=\bigoplus_{m\in\mathbb{Z}_{\ge0}} H^0\left(\frac{\mathring{\mathfrak{h}}_\mathbb{C}}{\mathring{\mathfrak{h}}_\mathbb{Z}+\tau\mathring{\mathfrak{h}}_\mathbb{Z}}, \mathcal{L}^{\otimes m}\right)$$

Let

$$\mathfrak{h}_\mathbb{C}=\mathbb{C}\delta\oplus\mathring{\mathfrak{h}}_\mathbb{C}\oplus\mathbb{C}\Lambda_0 \quad \text{and}$$

$$\Theta_{\bar\lambda+m\Lambda_0}=e^{m\Lambda_0}\sum_{\beta\in\mathring{\mathfrak{h}}_\mathbb{Z}} e^{\bar\lambda+m\beta-\frac{1}{2m}(\bar\lambda+m\beta|\bar\lambda+m\beta)\delta} \quad \text{for } \bar\lambda\in\mathring{\mathfrak{h}}^*_\mathbb{Z},\ m\in\mathbb{Z}_{>0}$$

in the ~~ring~~ "span" $\{e^\lambda \mid \lambda\in\mathfrak{h}_\mathbb{C}\}$ with $e^\lambda e^\mu=e^{\lambda+\mu}$.

Let

$$q=e^{-\delta}=e^{2\pi i\tau}$$

Th_0 = holomorphic functions on $\mathcal{H}^+$

Th_m has Th_0-basis $\{\Theta_{\bar\lambda+m\Lambda_0} \mid \bar\lambda\in\mathring{\mathfrak{h}}^*_\mathbb{Z} \bmod m\mathring{\mathfrak{h}}_\mathbb{Z}\}$

Then the $\mathbb{Z}_{\ge0}$-graded Th_0-algebra

$$Th=\bigoplus_{m\in\mathbb{Z}_{\ge0}} Th_m \quad \text{replaces } \mathcal{O}_{E^\ell} \quad \left(\begin{array}{c}\text{for all } \tau\\ \text{at once}\end{array}\right)$$

and Th-modules replace sheaves on E^ℓ.

On the right track?

(1) Chevalley-Shephard-Todd

(a) $H_G(pt) = H_T(pt) = S(\mathring{\mathfrak{h}}_{\mathbb{C}}^{*})^{W_0}$ is a polynomial ring.

(b) $H_T(pt)^{\det}$ is a free $H_T(pt)^{W_0}$-module of rank 1

$$\begin{aligned} H_T(pt)^{W_0} &\longrightarrow H_T(pt)^{\det} \\ f &\longmapsto e_\rho f \end{aligned} \qquad \text{with } e_\rho = \prod_{\alpha\in R^+} \alpha .$$

(c) $H_T(pt)$ is a free $H_T(pt)^{W_0}$-module of rank W_0.

(2) Weyl/Pittie-Steinberg/Demazure

(a) $K_G(pt) = K_T(pt)^{W_0}$ is a polynomial ring

(b) $K_T(pt)^{\det}$ is a free $K_T(pt)^{W_0}$-module of rank 1.

$$\begin{aligned} K_T(pt)^{W_0} &\xrightarrow{\sim} K_T(pt)^{\det} \\ f &\longmapsto a_\rho f \end{aligned} \qquad \text{with } a_\rho = \prod_{\alpha\in R^+} (1-e^{-\alpha})$$

(c) $K_T(pt)$ is a free $K_T(pt)^{W_0}$-module of rank $|W_0|$.

(3) Bernstein-Schwartzman/Kac-Peterson.

(a) $\overline{Th}^{W_0}$ is a polynomial ring

(b) $\overline{Th}^{\det}$ is a free $\overline{Th}^{W_0}$-module of rank 1

$$\begin{aligned} \overline{Th}^{W_0} &\xrightarrow{\sim} \overline{Th}^{\det} \\ f &\longmapsto A_\rho f \end{aligned} \qquad \text{with } A_\rho = \prod_{n\in\mathbb{Z}_{>0}} \Big((1-q^n) \prod_{\alpha\in R^+} (1-q^{n-1}e^{-\alpha})(1-q^n e^{\alpha})\Big)$$

(c) $\overline{Th}$ is a free $\overline{Th}^{W_0}$-module of rank $|W_0|$.

Flag varieties

A parabolic subgroup is $P_J \supseteq B$ with G/P_J a projective variety.

G/P_J are the partial flag varieties.

$$G = \bigsqcup_{w \in W_0} BwB \quad \text{and} \quad G = \bigsqcup_{u \in W^J} BuP_J$$

and

$$X_w = \overline{BwB} \text{ in } G/B$$
$$X_u^J = \overline{BuP_J} \text{ in } G/P_J$$

are the Schubert varieties

The T-fixed points in G/B are wB, $w \in W_0$.

$$\iota_w : pt \hookrightarrow G/B$$
$$* \longmapsto wB.$$

Example $G = GL_3(\mathbb{C})$, $W_0 = S_3 = \langle s_1, s_2 \mid s_i^2 = 1,\ s_1s_2s_1 = s_2s_1s_2 \rangle$

$$\begin{tikzcd}
& X_{s_1s_2s_1} = G/B & \\
X_{s_1s_2} \arrow[ur, hook] & & X_{s_2s_1} \arrow[ul, hook] \\
\mathbb{P}^1 \cong X_{s_1} \arrow[u, hook] \arrow[urr, hook] & & X_{s_2} \cong \mathbb{P}^1 \arrow[u, hook] \arrow[ull, hook] \\
& X_1 = pt \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& G/B \arrow[dl] \arrow[dr] & \\
\mathbb{P}^1 \cong G/P_2 \arrow[dr] & & G/P_1 = \mathbb{P}^1 \arrow[dl] \\
& G/G = pt &
\end{tikzcd}$$

where $P_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$ and $P_2 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$

Push-pull and Moment sections

The minimal $P_i \supseteq B$ and $\pi_i : G/B \to G/P_i$ give

$$T_i : H_T(G/B) \longrightarrow H_T(G/P_i) \longrightarrow H_T(G/B)$$

and $\sigma_w : X_w \hookrightarrow G/B$ give

$$\begin{aligned} (\sigma_w)_* : H_T(X_w) &\longrightarrow H_T(G/B) \\ 1 &\longmapsto [X_w] \end{aligned}$$

As T-modules

$$\mathfrak{g}/\mathfrak{b} = \bigoplus_{\alpha \in R^+} \mathbb{C}_{-\alpha} \quad \text{and} \quad \mathfrak{g}/\mathfrak{p}_i = \bigoplus_{\alpha \in R^+ - \{\alpha_i\}} \mathbb{C}_{-\alpha}$$

so that

$$T_i = \frac{1}{D_{\alpha_i}} + s_i \frac{1}{D_{\alpha_i}} \quad \text{where } D_{\alpha_i} \text{ is the Euler class of } \mathbb{C}_{-\alpha_i}$$

If $w = s_{i_1} \cdots s_{i_\ell}$ then

$$[X_w] = T_{i_\ell} \cdots T_{i_1} [X_1] \qquad \left(\begin{matrix}\text{BGG and} \\ \text{Demazure}\end{matrix}\right)$$

Moment sections:

$$\begin{aligned} \bigoplus_{w \in W_0} (\iota_w)^* : H_T(G/B) &\longrightarrow \bigoplus_{w \in W_0} H_T(pt) \\ f &\longmapsto (f_w) \end{aligned}$$

is a ring isomorphism onto its image with

$$(fg)_w = f_w g_w \quad \text{and} \quad (uf)_w = f_{wu^{-1}} \quad \text{for } w \in W_0.$$

Then

$$[X_1]_w = \begin{cases} 0, & \text{if } w \neq 1, \\ \prod_{\alpha \in R^+} D_\alpha, & \text{if } w = 1. \end{cases}$$

O'SULLIVAN CONFERENCE PROGRAM

1. Schedule of talks

	Friday 2/9	Saturday 3/9	Sunday 4/9
9:30-10:30am	Chuck Weibel	Maksym Fedorchuk	Thomas Geisser
11:00-12:00am	Bruno Kahn	Vadusevan Srinivas	Amalendu Krishna
1:30-2:30pm	Sinan Unver	Arun Ram	Marco Schlichting
2:45-3:45pm	Shun-ichi Kimura	Anthony Henderson	Geordie Williamson
4:15-5:15pm	Ross Street	Yves André	Vladimir Guletskii

2. Breaks

There will be morning breaks from 10:30 to 11:00am, afternoon breaks from 3:45 to 4:15pm, with refreshments served in the common room. Lunchtime will be from 12:00 to 1:30.

3. Social activities

(i) Welcome reception starting at 5:30pm on Friday, in the common room.
(ii) Conference dinner starting at 6:30pm on Saturday, at the Jewel of India restaurant at 24 West Row (in the Melbourne Building). The conference dinner is free for registered participants, $30 per unregistered person.
(iii) A farewell party for the participants starting at 6:30pm on Sunday at Amnon Neeman's house (3 Elimatta Street, Reid).

4. Restaurant recommendations

Spicy Ginger Café, 25 Childer Street (Szechuan).

Jewel of India, 24 West Row, Melbourne Building (Indian).

Iori, 41 East Row, Sydney Building (Japanese).

Lemon Grass, 65 London Circuit, Melbourne Building (Thai).

Teatro Vivaldi, Union Court, on campus (International).