Elliptic Schubert Calculus, Isaac Newton Institute 16 Sept. 2011 (1)

Schubert calculus: Cohomology of G/B

G connected reductive alg. group$/\mathbb{C}$.
$\cup|$
B Borel subgroup
$\cup|$
T maximal torus

$$GL_n(\mathbb{C}) \supseteq \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \supseteq \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

$W_0 = N(T)/T$ is the Weyl group.

A parabolic subgroup of G is $P_J \supseteq B$ with

G/P_J is a projective variety (the partial flag varieties)

G/B is the flag variety.

$$G = \bigsqcup_{w\in W_0} BwB \quad \text{and} \quad G = \bigsqcup_{u\in W^J} BuP_J$$ (Bruhat decomposition)

where $W_J = \{v \in W_0 \mid v \in P_J\}$ and

$W^J = \{$ coset representatives u of cosets in $W/W_J\}$

$X_w = \overline{BwB}$ in G/B

$X_u^J = \overline{BuP_J}$ in G/P_J are the Schubert varieties.

The T-fixed points in G/P_J are uP_J, $u \in W^J$

in G/B are wB, $w\in W_0$.

So

$$\pi_J : G/B \to G/P_J \qquad \sigma_w : X_w \hookrightarrow G/B, \qquad \iota_w : pt \to G/B$$
$$gB \mapsto gP_J \qquad\qquad\qquad\qquad\qquad * \mapsto wB.$$

Generalised cohomologies (Adams / May / Bressler-Evens)

Examples

$H_T(G/B)$ equivariant cohomology

(... Bernstein-Gelfand-Gelfand, ...)

$K_T(G/B)$ equivariant K-theory

(... Demazure, Kostant-Kumar, ...)

$Ell_T(G/B)$ equivariant Elliptic cohomology

(... Grojnowski; Ginzburg-Kapranov-Vasserot, Ando, ...)

$\Omega_T(G/B)$ equivariant complex cobordism

(... Calmès-Petrov-Zainoulline, Krichenko-Krishna, ...)

Axioms / Tools (0) Normalization $H_T(pt)$

(1) Products, smashes, suspensions $H_{G\times K}(M\times N)$

(2) Functoriality

If $f: X \to Y$ then $f^*: H_T(Y) \to H_T(X)$

(3) Thom isomorphism / orientability

If $f: X \to Y$ then try to make $f_*: H_T(X) \to H_T(Y)$

(4) Change of groups: If $\varphi: G \to K$ then try to make

$X_\varphi: H_G \to H_K$ and $X^\varphi: H_K \to H_G$.

Normalization, Weyl characters, Borel picture

(H) ...

(K) $\quad K_T(pt) = \mathbb{C}[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}], \qquad K_T(pt)^{W_0} = K_G(pt)$

$K_T(pt) = \mathrm{span}\{e^\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\}$ with $e^\lambda e^\mu = e^{\lambda+\mu}$

$K_G(pt)$ is a polynomial ring.

$$K_T(pt)^{W_0} \xrightarrow{\sim} K_T(pt)^{\det} \quad \text{as } K_T(pt)^{W_0}\text{-modules}$$
$$f \longmapsto a_\rho f$$

Weyl character $\quad s_\lambda \longleftrightarrow a_{\lambda+\rho} = \sum_{w \in W_0} \det(w) e^{w(\lambda+\rho)}$

with $\quad a_\rho = e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha}) \qquad$ (Euler class of $pt^{\mathfrak{g}/\mathfrak{b}}$)

Then
$$K_T(G/B) = K_T(pt) \otimes_{K_G(pt)} K_T(pt)$$
$$= \frac{\mathbb{C}[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}, Y_1^{\pm 1}, \ldots, Y_\ell^{\pm 1}]}{\langle f(X_1, \ldots, X_\ell) = f(Y_1, \ldots, Y_\ell) \text{ for } f \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}]^{W_0} \rangle}$$

(E) $\quad Ell_T(pt) = \mathcal{O}_{E^\ell} \quad$ and $\quad Ell_G(pt) = \mathcal{O}_{E^\ell / W_0}$

with $E = \dfrac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}}$ is an elliptic curve, $\tau \in \mathcal{H}^+$

$E^\ell = \dfrac{\mathfrak{h}^*_{\mathbb{C}}}{\mathfrak{h}^*_{\mathbb{Z}} + \tau \mathfrak{h}^*_{\mathbb{Z}}}$ is an abelian variety

Replacing $\mathcal{O}_{E\ell}$ by Th

$\mathfrak{h}_{\mathbb{Z}}$ comes with a positive. def. symm form $(\,|\,)$ which provides an ample line bundle $\mathcal{L}$ and

$${}^{\tau}\mathrm{Th} = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \left(\frac{\mathring{\mathfrak{h}}_{\mathbb{C}}}{\mathring{\mathfrak{h}}_{\mathbb{Z}}+\tau\mathring{\mathfrak{h}}_{\mathbb{Z}}}, \mathcal{L}^{\otimes m}\right)$$

Let

$$\mathfrak{h}_{\mathbb{C}} = \mathbb{C}\delta \oplus \mathring{\mathfrak{h}}_{\mathbb{C}} \oplus \mathbb{C}\Lambda_0 = \left\{ \begin{pmatrix} \underline{a} \\ \bar\lambda \\ \underline{m} \end{pmatrix} \,\middle|\, \begin{matrix} a\in\mathbb{C} \\ \bar\lambda\in\mathring{\mathfrak{h}}_{\mathbb{C}} \\ m\in\mathbb{C} \end{matrix} \right\}$$

and for $p\in\mathring{\mathfrak{h}}_{\mathbb{Z}}$ define $t_p : \mathfrak{h}_{\mathbb{C}} \to \mathfrak{h}_{\mathbb{C}}$ by

$$t_p(\lambda) = t_p(a\delta+\bar\lambda+m\Lambda_0) = \begin{pmatrix} 1 & -p & -\frac{1}{2}(p|p) \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{a} \\ \bar\lambda \\ \underline{m} \end{pmatrix} = \left(a-(\bar\lambda|p)-\tfrac{1}{2}(mp|p)\right)\delta + \bar\lambda + mp + m\Lambda_0$$

and

$$\Theta_{\bar\lambda+m\Lambda_0} = e^{-\frac{1}{2m}(\lambda|\lambda)\delta} \sum_{p\in\mathring{\mathfrak{h}}_{\mathbb{Z}}} e^{t_p(\bar\lambda+m\Lambda_0)} = e^{m\Lambda_0} \sum_{p\in\mathring{\mathfrak{h}}_{\mathbb{Z}}} e^{\bar\lambda+mp}\, q^{\frac{1}{2m}\|\bar\lambda+mp\|^2 \delta}$$

where $q = e^{-\delta} = e^{2\pi i\tau}$. Then

$$\mathrm{Th} = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathrm{Th}_m \quad \text{is a graded } \mathrm{Th}_0\text{-algebra}$$

where Th_0 = holomorphic functions on $\mathcal{H}^+$

Th_m has Th_0-basis $\{\Theta_{\bar\lambda+m\Lambda_0} \mid \bar\lambda\in\mathring{\mathfrak{h}}_{\mathbb{Z}}^{*} \bmod m\mathring{\mathfrak{h}}_{\mathbb{Z}}\}$

Completion of the Borel picture of $Ell_T(G/B)$

W_0 acts on $\hat{\mathfrak{h}}$

$$w(a\delta+\bar\lambda+m\Lambda_0)=\left(\begin{array}{c|c|c}1&0&0\\ \hline 0&w&0\\ \hline 0&0&1\end{array}\right)\begin{pmatrix}a\\ \hline \bar\lambda\\ \hline m\end{pmatrix}=a\delta+w\bar\lambda+m\Lambda_0$$

and

$$we^{\lambda}=e^{w\lambda}\quad\text{and}\quad w\Theta_\lambda=\Theta_{w\lambda}.$$

Then

$\hat{\mathfrak{h}}^{W_0}$ is a polynomial ring (Looijenga, Bernstein-Schwartzman)

and Kac-Peterson explain

$$\hat{\mathfrak{h}}^{W_0}\xrightarrow{\sim}\hat{\mathfrak{h}}^{\det}$$

$$f\longmapsto a_{\hat\rho}f$$

Weyl-Kac character for $\hat{\mathfrak{g}}$ $\quad s_\lambda\longmapsto a_{\lambda+\hat\rho}=(\mathrm{const})\sum_{w\in W_0}\det(w)\,\Theta_{w(\lambda+\hat\rho)}$

where $\hat\rho=\rho+h\Lambda_0$, where h is the dual Coxeter number

$\hat{\mathfrak{h}}^{\det}=\{f\in\hat{\mathfrak{h}}\mid wf=\det(w)f \text{ for } w\in W_0\}$.

$$a_{\hat\rho}=(\mathrm{const})\prod_{k\in\mathbb{Z}_{>0}}\Big((1-q^k)^\ell\prod_{\alpha\in R^+}(1-q^{k-1}e^{-\alpha})(1-q^ke^{\alpha})\Big)$$ (elliptic Euler class for $pt^{\hat{\mathfrak{h}}}$)

Then $Ell_T(pt)=\hat{\mathfrak{h}}$ and $Ell_G(pt)=\hat{\mathfrak{h}}^{W_0}$

$$Ell_T(G/B)=\hat{\mathfrak{h}}\otimes_{\hat{\mathfrak{h}}^{W_0}}\hat{\mathfrak{h}}$$

$$\Big\uparrow \pi_J^*$$

$$Ell_T(G/P_J)=\hat{\mathfrak{h}}\otimes_{\hat{\mathfrak{h}}^{W_0}}\hat{\mathfrak{h}}^{W_J}$$

Moment sections

Consider

$$\bigoplus_{w \in W_0} \tilde{K}_h = \{ z = (z_w) \mid z_w \in \tilde{K}_h \}$$

with $\tilde{K}_h \otimes_{\tilde{K}_h^{W_0}} \tilde{K}_h$-action

$$(f(x) g(q) z)_w = (wf)(y) g(y) z_w$$

and W_0-action and product given by

$$(vz)_w = z_{wv^{-1}} \quad \text{and} \quad (z\gamma)_w = z_w \gamma_w$$

Let

$$T_{s_i} = (1 + s_i) \frac{1}{D_{\alpha_i}} \quad \text{and} \quad [X_1]_w = \begin{cases} a_{\hat{\rho}}, & \text{if } w = 1 \\ 0, & \text{if } w \neq 1 \end{cases}$$

where

$$D_{\alpha_i} = (\text{const}) \prod_{k \in \mathbb{Z}_{>0}} (1 - q^{k-1} e^{-\alpha_i})(1 - q^k e^{\alpha_i}) \quad \text{and}$$

$$a_{\hat{\rho}} = (\text{const}) \prod_{k \in \mathbb{Z}_{>0}} \Big((1-q^k)^\ell \prod_{\alpha \in R^+} (1 - q^{k-1} e^{-\alpha_i})(1 - q^k e^{\alpha_i}) \Big)$$

Define, for $w = s_{i_1} \cdots s_{i_\ell}$ a reduced decomposition

$$[X_{\vec{w}}] = T_{i_1} \cdots T_{i_\ell} [X_1]$$

These "Schubert classes" are the images (pushforwards) of the Bott-Samelson resolutions

$$P_{i_1} \times^B P_{i_2} \times^B \cdots \times^B P_{i_\ell} / B \longrightarrow G/B$$

Isaac Newton Institute
for Mathematical Sciences

Workshop Programme
for period 12-16 September

Perspectives in Algebraic Lie Theory

12-16 September
Timetable

Monday 12 September

09:00-10:00	Registration	
10:00-11:00	**Brundan, J** *(Oregon)* Koszulity of the walled Brauer algebra	Sem 1
11:00-11:30	Coffee	
11:30-12:30	**Tanisaki, T** *(Osaka City)* Differential operators on quantized flag manifolds	Sem 1
12:30-13:30	Lunch	
14:00-15:00	**Rumynin, D** *(Warwick)* Double Cells and Representations	Sem 1
15:00-15:30	Tea	
15:30-16:30	**Webster, B** *(Oregon)* Categorification, canonical bases and knot invariants	Sem 1
16:30-17:00	Drinks reception	

Tuesday 13 September

09:00-10:00	**Kessar, R** *(Aberdeen)* Quasi-isolated blocks of exceptional groups	Sem 1
10:00-11:00	**Williamson, G** *(Oxford)* Some applications of parity sheaves	Sem 1
11:00-11:30	Coffee	
12:30-13:30	Lunch	
14:00-15:00	**Robinson, G** *(Aberdeen)* Some remarks on endomorphism rings and realizability questions	Sem 1
15:00-15:30	Tea	
15:30-16:30	**Erdmann, K** *(Oxford)* Lie powers for general linear groups and Lie modules for symmetric groups	Sem 1

Wednesday 14 September

09:00-10:00	**Opdam, E** *(Amsterdam)* Extensions of tempered representations and R-groups	Sem 1
10:00-11:00	**Soergel, W** *(Albert-Ludwigs-Universität Freiburg)* Koszul duality in positive characteristic	Sem 1
11:00-11:30	Coffee	
11:30-12:30	**Premet, A** *(Manchester)* Hesselink stratification of nullcones and base change	Sem 1
12:30-13:30	Lunch	
14:00-15:00	**Lusztig, G** *(MIT)* On disconnected reductive groups	Sem 1

15:00-15:30	Tea	
19:30-22:00	Conference dinner at Peterhouse	

Thursday 15 September

09:00-10:00	**Goodwin, S** *(Birmingham)*	
	Representations of finite W-algebras associated to classical Lie algebras	Sem 1
10:00-11:00	**Ostrik, V** *(Oregon)*	
	Classification of finite dimensional irreducible modules over finite W-algebras	Sem 1
11:00-11:30	Coffee	
11:30-12:30	**Arakawa, T** *(Kyoto)*	
	Localization of affine W-algebras at the critical level	Sem 1
12:30-13:30	Lunch	
14:00-15:00	**Fiebig, P** *(Universität Erlangen-Nürnberg)*	
	On critical level representations of affine Kac-Moody algebras	Sem 1
15:00-15:30	Tea	
15:30-16:00	Tea	

Friday 16 September

09:00-10:00	**Lauda, A** *(Columbia)*	
	The Odd NilHecke algebra	Sem 1
10:00-11:00	**Vasserot, E** *(Université Paris 7 - Denis-Diderot)*	
	Categorification and finite dimensional modules of DAHA	Sem 1
11:00-11:30	Coffee	
11:30-12:30	**Juteau, D** *(CNRS - Université de Caen Basse-Normandie)*	
	Singularities in nilpotent cones of exceptional type	Sem 1
12:30-13:30	Lunch	
14:00-15:00	**Ram, A** *(Melbourne)*	
	Elliptic Schubert calculus	Sem 1
15:00-15:30	Tea	

Perspectives in Algebraic Lie Theory
12-16 September
Gerhard Roehrle

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