Talk in working seminar: 28.10.2011

Schubert calculus: Cohomology of G/B

G/B is the flag variety.

G connected reductive alg. gp / $\mathbb{C}$ $\qquad GL_n(\mathbb{C})$

$\cup|$ $\qquad \cup|$

B Borel subgroup $\qquad \left\{\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}\right\}$

$\cup|$ $\qquad \cup|$

T maximal torus $\qquad \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$

$W_0 = N(T)/T$ is the Weyl group.

$\mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}(T, \mathbb{C}^\times)$ and $\mathfrak{h}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^\times, T)$

A parabolic subgroup of G is $P_J \supseteq B$ with

G/P_J is a projective variety (the partial flag varieties)

(Bruhat decomposition) $\quad G = \bigsqcup_{w \in W_0} BwB$ and $G = \bigsqcup_{u \in W^J} BuP_J$

where $W_J = \{ v \in W_0 \mid vT \in P_J \}$ and

$W^J = \{$coset representatives of cosets in $W_0/W_J\}$.

$X_w = \overline{BwB}$ in G/B

$X_u^J = \overline{BuP_J}$ in G/P_J $\quad$ are the Schubert varieties

The T-fixed points

in G/B are $\{wB \mid w \in W_0\}$

in G/P_J are $\{uP_J \mid u \in W^J\}$

Let $P_1, \ldots, P_n$ be the minimal parabolic subgroups ($P_i \neq B$ and $P_i = P_{\{i\}}$). Then

$$W_i = W_{\{i\}} = \{1, s_i\} \quad \text{and} \quad s_1, s_2, \ldots, s_n$$

are the simple reflections in W_0.

Proposition W_0 is generated by $s_1, \ldots, s_n$.

Let $w \in W_0$ and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w. The Bott tower or Bott-Samelson variety corresponding to $s_{i_1} \cdots s_{i_\ell}$ is

$$\begin{array}{ccccc} P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_\ell}/B & \longrightarrow & X_w & \hookrightarrow & G/B \\ (x_{i_1}(c_1)s_{i_1}, \ldots, x_{i_\ell}(c_\ell)s_{i_\ell}B) & \longmapsto & x_{i_1}(c_1)s_{i_1} \cdots x_{i_\ell}(c_\ell)s_{i_\ell}B. & & \end{array}$$

Provides

$\pi_J : G/B \to G/P_J$ $\qquad$ $\rho_{i_1 \cdots i_\ell} : P_{i_1} \times_B \cdots \times_B P_{i_\ell}/B \to G/B$

$j_w : X_w \hookrightarrow G/B$ $\qquad$ $j_u^J : X_u^J \hookrightarrow G/P_J$

$$\begin{array}{rccc} \iota_w : & pt & \hookrightarrow & G/B \\ & * & \longmapsto & wB \end{array} \qquad \begin{array}{rccc} \iota_u^J : & pt & \hookrightarrow & G/P_J \\ & * & \longmapsto & uP_J. \end{array}$$

Normalization, Weyl characters, Borel picture

K-theory

(a) $K_T(pt) = \mathbb{C}[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}] = \mathrm{span}\{e^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*\}$

with $e^\lambda e^\mu = e^{\lambda+\mu}$.

(b) $K_G(pt) = K_T(pt)^{W_0}$ is a polynomial ring.

(c) $$K_T(pt)^{W_0} \xrightarrow{\sim} K_T(pt)^{\det} \quad \text{as } K_T(pt)\text{-modules.}$$
$$f \longmapsto a_\rho f$$
$$s_\lambda \longmapsto a_{\lambda+\rho} = \sum_{w \in W_0} \det(w) e^{w(\lambda+\rho)}$$

with $a_\rho = e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha})$ (Euler class of $pt^{G/B}$).

(d) $$K_T(G/B) = K_T(pt) \otimes_{K_G(pt)} K_T(pt)$$
$$= \frac{\mathbb{C}[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}, Y_1^{\pm 1}, \ldots, Y_\ell^{\pm 1}]}{\langle f(X_1, \ldots X_\ell) = f(Y_1, \ldots, Y_\ell) \text{ for } f \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}]^{W_0} \rangle}$$

(e) $K_T(pt)$ is a free $K_T(pt)^{W_0}$-module with basis $\{T_w [\mathcal{O}_{X_1}] \mid w \in W_0\}$

Generalised-cohomologies (Adams/May/Bressler-Evens).

Examples

$H_T(G/B)$ equivariant cohomology

(... Bernstein-Gelfand-Gelfand, ...)

$K_T(G/B)$ equivariant K-theory

(... Demazure, Kostant-Kumar, ...)

$Ell_T(G/B)$ equivariant Elliptic cohomology

(... Grojnowski; Ginzburg-Kapranov-Vasserot, Ando, ...)

$\Omega_T(G/B)$ equivariant complex cobordism

(... Calmès-Petrov-Zainoulline, Kiritchenko-Krishna, ...)

Axioms/tools (0) Normalization $H_T(pt)$

(1) Products, smashes, suspensions $H_{G\times K}(M\times N)$

(2) Functoriality/pullbacks

If $f: X \to Y$ then $f^*: H_T(Y) \to H_T(X)$

(3) Thom isomorphism/orientability

If $f: X \to Y$ then try to make $f_*: H_T(X) \to H_T(Y)$

(4) Change of groups: If $\varphi: G \to K$ then try to make

$\chi_\varphi: H_G \to H_K$ and $\chi^\varphi: H_K \to H_G$.

Let R be a ring. A <u>formal</u> ~~ring~~ <u>group law</u> <u>over R</u> is $F \in R[[x,y]]$ such that

$$F(x,y) = x + y + \sum_{i,j \in \mathbb{Z}_{\geq 1}} c_{ij} x^i y^j \qquad \text{and} \quad F(x,0) = F(0,x) = x,$$

$$F(x, F(y,z)) = F(F(x,y),z). \text{ and } F(x,y) = F(y,x).$$

The <u>Lazard</u> <u>ring</u> is the ring $\mathbb{L}$ generated by $\{c_{ij} \mid i,j \in \mathbb{Z}_{\geq 0}\}$ with

$$c_{00} = 0, \ c_{10} = 1, \ c_{01} = 1, \quad c_{0i} = c_{i0} = 0 \text{ for } i > 1,$$

and $F(x, F(y,z)) = F(F(x,y),z)$ and $F(x,0) = F(0,x) = x$ and $F(x,y) = F(y,x)$ where $F = x + y + \sum_{i,j} c_{ij} x^i y^j$.

<u>Lazard's</u> <u>theorem</u> $\mathbb{L} = \mathbb{Z}[c_{ij}]$ with $\deg(c_{ij}) = 2(i+j-1)$

<u>Quillen's</u> <u>theorem</u> $\Omega(pt) = \mathbb{L}$.

<u>Quillen</u>$_T$ $\Omega_T(pt) = \mathbb{L}[[x_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}]]$ with

$$x_{\lambda+\mu} = x_\lambda +_F x_\mu = F(x_\lambda, x_\mu)$$