20 years trekking on the LS path: Talk of Arun Ram at Seshadri 80th birthday conference 23-27 January 2012, CMI, Chennai.

Before the trek

1987 Weyl, The classical groups

1988 Macdonald, Symmetric Functions and Hall polys

1990 Littelmann, A generalization of the LR rule

Lakshmibai-Seshadri, Geometry of G/P-V.

1990 Seshadri, Introduction to Standard Monomial Theory, Brandeis Univ. Lect. Notes.

1992 Lakshmibai-Seshadri, Standard Monomial Theory Hyderabad conference volume

Lakshmibai-Conjecture-Littelmann email "I proved your conjecture"

$$\mathrm{char}(H^0(G/B, \mathcal{L}_\lambda)) = \frac{\sum_{w\in W_0} \det(w) e^{w(\lambda+\rho)}}{e^\rho \prod_{\alpha\in R^+}(1-e^{-\alpha})} = \sum_{p\in B(\lambda)} e^{\mathrm{end}(p)}$$

Views from the LS path

Crystals, Affine Hecke algebra

Schubert calculus, Loop groups

... the horizon.

Data

G = complex reductive algebraic group

$\cup\!|$

B = Borel subgroup

$\cup\!|$

T maximal torus

The Weyl group is $W_0 = N(T)/T$ acts on

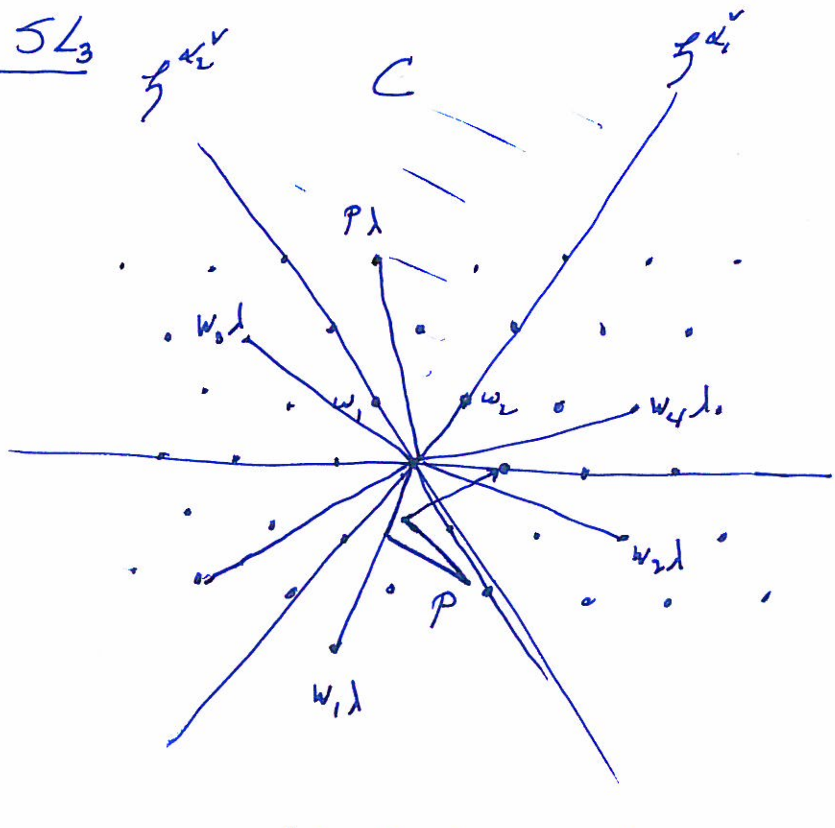
$$\mathfrak{h}^*_{\mathbb{Z}} = \mathrm{Hom}(T, \mathbb{C}^\times) \quad \text{and} \quad \mathfrak{h}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^\times, T)$$

C is a fundamental chamber for W_0-action on $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}}$

W_0 is generated by $s_1, s_2, \ldots, s_n$

reflections in the walls $\mathfrak{h}^{\alpha_1^\vee}, \ldots, \mathfrak{h}^{\alpha_n^\vee}$ of C.

Type SL_3



$$p = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

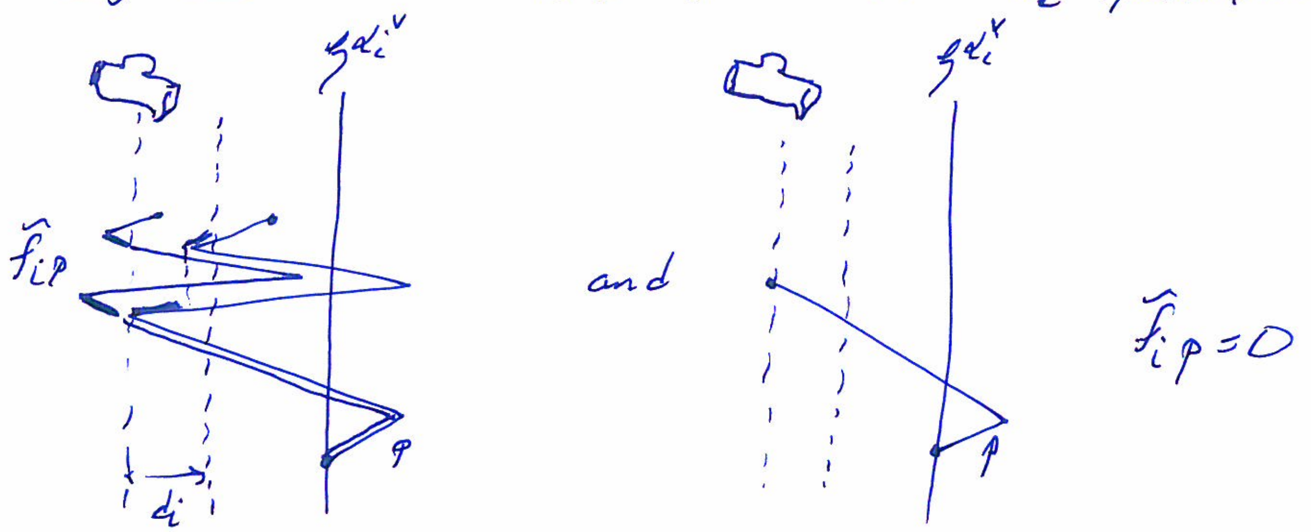
initial direction = $\iota(p)$ (= w_1) $\qquad$ $\varphi(p)$ = final direction (= w_4)

G/B is the *flag variety* and

$$G = \bigsqcup_{w \in W_0} BwB \qquad \text{and} \qquad X_w = \overline{BwB} \quad \text{in } G/B$$

are the *Schubert varieties*.

Crystals For $i = 1, 2, \ldots, n$ define $\tilde{f}_i : \{\text{paths}\} \to \{\text{paths}\} \cup \{0\}$



For $\lambda \in \mathfrak{h}^*_{\mathbb{Z}} \cap (C - \rho)$ let $p_\lambda : [0,1] \to \mathfrak{h}^*_{\mathbb{R}}$ with $p_\lambda(1) = \lambda$ and $p_\lambda([0,1]) \subseteq C - \rho$. Let

$$B(\lambda) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_m} p_\lambda \mid m \in \mathbb{Z}_{\geq 0},\ 1 \leq i_1, \ldots, i_m \leq n \}$$

Let $w = s_{i_1} \cdots s_{i_\ell}$ be minimal length. Then

$$\mathrm{char}(H^0(X_w, \mathcal{L}_\lambda)) = T_{i_1} \cdots T_{i_\ell} e^\lambda = \sum_{p \in B(\lambda)_{\leq w}} e^{\mathrm{end}(p)}$$

where

$$\mathcal{L}_\lambda = \begin{array}{c} G \times_B \mathbb{C}_\lambda \\ \downarrow \\ G/B \end{array}, \qquad T_i = e^{-\rho} \frac{1}{1 - e^{-\alpha_i}} (1 - s_i) e^{\rho} \quad \text{and}$$

$$B(\lambda)_{\leq w} = \{ p \in B(\lambda) \mid \mathrm{in}(p) \leq w \}.$$

Schubert calculus

The operators

$$T_i : Rep(T) \to Rep(T) \quad \text{and} \quad X^\mu : Rep(T) \to Rep(T)$$
$$f \mapsto e^\mu f$$

provide a representation of the nil affine Hecke algebra

$H(0)$ has generators T_i and X^μ

relations

$$T_i^2 = T_i \quad \text{and} \quad T_i T_j T_i \cdots = T_j T_i T_j \cdots$$

$$T_i X^\mu = X^{s_i\mu} T_i + \frac{X^\mu - X^{s_i\mu}}{1 - X^{-\alpha_i}}$$

Theorem Let $\lambda \in \mathfrak{h}^*_{\mathbb{Z}} \cap (C-\rho)$ and $w \in W_0$.

(a) In $H(0)$,

$$T_{w^{-1}} X^\mu = \sum_{p \in B(\lambda)_{\le w}} X^{end(p)} T_{\varphi(p)^{-1}}$$

(b) In $K_T(G/B)$,

$$[\mathcal{O}_{X_w}][\mathcal{L}_\lambda] = \sum_{p \in B(\lambda)_{\le w}} e^{end(p)} [\mathcal{O}_{X_{\varphi(p)}}]$$

(c) The T-equivariant sheaf on G/B, $\mathcal{F} = \mathcal{O}_{X_w} \otimes \mathcal{L}_\lambda$

has

$$\mathcal{F} \supseteq \mathcal{F}^{(1)} \supseteq \mathcal{F}^{(2)} \supseteq \cdots$$

so that the quotients are $e^{end(p)} \otimes \mathcal{O}_{X_{\varphi(p)}}$.

Loop groups

$$\begin{array}{ccc} G = G(\mathbb{C}((t))) & & \\ \cup\!| & & \\ K = G(\mathbb{C}[[t]]) & \xrightarrow[\Phi]{t=0} & G(\mathbb{C}) \\ \cup\!| & & \cup\!| \\ I = \Phi^{-1}(B) & \longrightarrow & B \end{array}$$

G is presented by generators

$$x_\alpha(f), \quad x_{-\alpha}(f) \quad \text{and} \quad h_{\lambda^\vee}(g)$$

$\alpha \in R^+$, $\lambda^\vee \in \mathfrak{h}_\mathbb{Z}$, $f \in \mathbb{C}((t))$, $g \in \mathbb{C}((t))^\times$ with

Steinberg-Tits relations

Let

$$U^- = \langle x_{-\alpha}(f) \mid \alpha \in R^+, f \in \mathbb{C}((t)) \rangle = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}$$

The <u>affine</u> Weyl group is

$$W = W_0 \ltimes \mathfrak{h}_\mathbb{Z} = \{ w t_{\lambda^\vee} \mid w \in W_0, \ \lambda^\vee \in \mathfrak{h}_\mathbb{Z} \}$$

where $t_{\lambda^\vee} = h_{\lambda^\vee}(t^{-1})$. Then

$$G = \bigsqcup_{w \in W_0} I w I, \qquad G = \bigsqcup_{v \in W} U^- v I$$

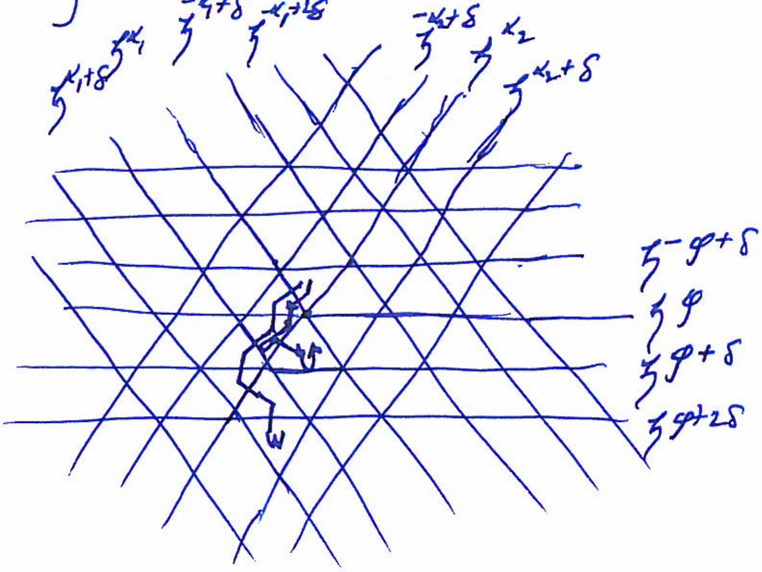
$$G = \bigsqcup_{\lambda^\vee \in \mathfrak{h}_\mathbb{Z}/W_0} K t_{\lambda^\vee} K \qquad G = \bigsqcup_{\mu^\vee \in \mathfrak{h}_\mathbb{Z}} U^- t_{\mu^\vee} K$$

The <u>Mirković-Vilonen</u> <u>intersections</u> are

$$IwI \cap U^- v I \quad \text{and} \quad K t_{\lambda^\vee} K \cap U^- t_{\mu^\vee} K.$$

Theorem

$$\left\{\begin{matrix}\text{labeled positively folded walks}\\ \text{of type } \vec{w} \text{ with end } v\end{matrix}\right\} \xleftrightarrow{1-1} IwI \cap U^{-}vI$$



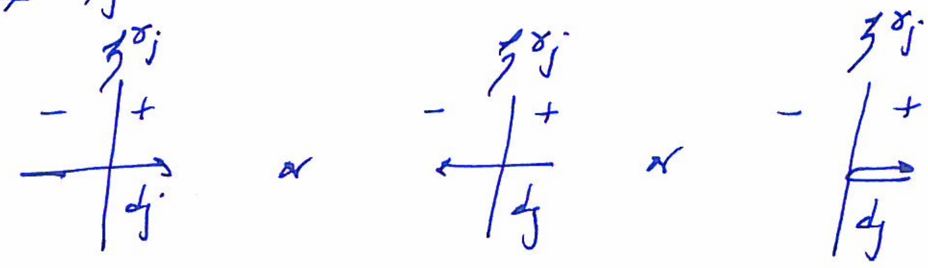
First note

$$W \xleftrightarrow{1-1} \{\text{alcoves}\}$$

$$x_{\alpha+k\delta}(c) = x_\alpha(ct^k)$$

Fix $\vec{w} = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ minimal length to w.

A step p_j is



$d_j \in \mathbb{C}$ $\qquad$ $d_j = 0$ $\qquad$ $d_j \in \mathbb{C}^\times$

where the periodic orientation is given by

(a) hyperplanes through 0 have 1 on positive side

(b) parallel hyperplanes have parallel orientation.

$$p = (p_1, p_2, \ldots, p_\ell) \longmapsto x_{\gamma_1}(d_1)x_{\gamma_2}(d_2)\cdots x_{\gamma_\ell}(d_\ell)vI.$$

... the horizon

Preprojective variety

Λ_p = preprojective cycles

Quiver variety or KLR modules

L_p = simple module

Loop Grassmannian

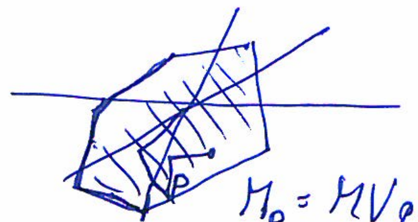
Z_p = MV cycle

shuffle algebra

char(Λ_p) semicanonical basis

char(L_p) canonical basis

char(Z_p) MV basis



M_p = MV polytope = Convex hull of $p([0,1])$ as p_λ varies.