~~Combinatorics and Growth in Chevalley Groups~~

Reflection groups

Talk at AMSI Workshop, Bridging the finite and infinite, Wollongong

2 February 2011

A $\underline{\mathbb{Z}\text{-reflection group}}$ is a pair $(W_0, \mathfrak{h}_{\mathbb{Z}})$ with

(a) $\mathfrak{h}_{\mathbb{Z}}$ a free $\mathbb{Z}$-module

(b) W_0 a finite subgroup of $GL(\mathfrak{h}_{\mathbb{Z}})$ generated by reflections

A $\underline{\text{reflection}}$ is $s \in GL(\mathfrak{h}_{\mathbb{C}})$ conjugate to

$$\begin{pmatrix} \xi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad \text{with } \xi \neq 1.$$

Let C be a fundamental chamber for the W_0-action on $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^*$. Let $s_1, \ldots, s_n$ be the reflections in the walls $\mathfrak{h}^{\alpha_1}, \ldots, \mathfrak{h}^{\alpha_n}$ of C.

__Theorem__ W_0 is presented by generators $s_1, \ldots, s_n$ and relations

$$s_i^2 = 1 \quad \text{and} \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}.$$

where $\pi/m_{ij} = \mathfrak{h}^{\alpha_i^\vee} \measuredangle \mathfrak{h}^{\alpha_j^\vee}$.

Let R^+ be an index set for reflections in W_0

$R = \{\alpha, -\alpha \mid \alpha \in R^+\}$.

Chevalley groups

Let $\mathbb{F}$ be a field (or commutative ring)

The Chevalley group corresponding to $(W_0, \mathfrak{h}_{\mathbb{Z}})$ and $\mathbb{F}$ is given by generators

$$x_\alpha(f) \text{ and } h_{\lambda^\vee}(g), \qquad \alpha \in R,\ \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}},\ f \in \mathbb{F},\ g \in \mathbb{F}^\times$$

with relations

- G contains a copy of $SL_2(\mathbb{F})$ for each $\alpha \in R^+$

$$SL_2(\mathbb{F}) \cong \langle x_\alpha(f), x_{-\alpha}(f) \mid f \in \mathbb{F} \rangle$$

- G has a symmetry under the subgroup

N generated by $h_{\lambda^\vee}(g)$ and $n_\alpha(g) = x_\alpha(g)\, x_{-\alpha}(-g^{-1})\, x_\alpha(g)$

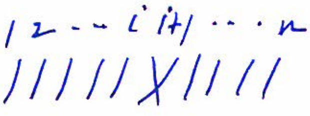
(relations like $n_\alpha(g)\, x_\beta(f)\, n_\alpha(g)^{-1} = x_{s_\alpha \beta}(\varepsilon_{\alpha\beta}\, g^{-\langle \beta, \alpha^\vee \rangle} f)$).

Example Type GL_n

$W_0 = S_n$ acts on $\mathfrak{h}_{\mathbb{Z}} = \sum_{i=1}^{n} \mathbb{Z} \varepsilon_i^\vee$

by permuting $\varepsilon_1^\vee, \ldots, \varepsilon_n^\vee$. If

$$C = \{ \lambda = \lambda_1 \varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,\ \lambda_i \in \mathbb{R} \}$$

then $s_i = (i, i+1) =$  for $i = 1, 2, \ldots, n-1$

and $R^+ = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n \}$.

$GL_n(\mathbb{F})$ is generated by

$$x_{ij}(f) = \begin{array}{c c} & j \\ i & \begin{pmatrix} 1 & & & \\ & \ddots & f & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \end{array} \qquad x_{ji}(f) = \begin{array}{c c} & j \\ i & \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & f & \ddots & \\ & & & 1 \end{pmatrix} \end{array}$$

for $1 \le i < j \le n$ and $f \in \mathbb{F}$.

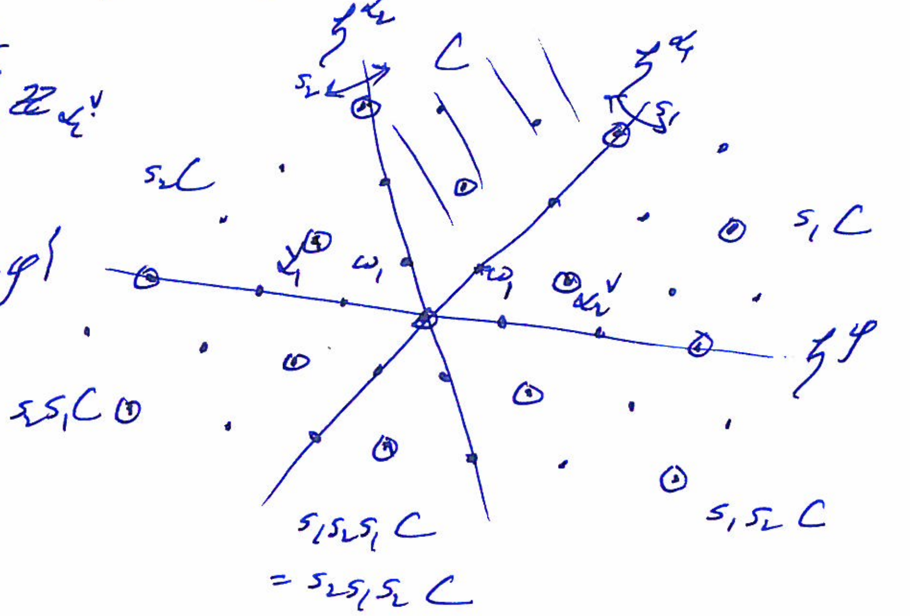
and

$$h_{\varepsilon_i^\vee}(g) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & g & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad \text{for } i = 1, 2, \ldots, n,\ g \in \mathbb{F}^\times$$

__Example__ __Type SL_3__

$$\mathfrak{h}_{\mathbb{Z}} = \sum_{i=1}^{2} \mathbb{Z}\alpha_i^\vee$$

$$R^+ = \{\alpha_1, \alpha_2, \varphi\}$$



and $SL_3(\mathbb{F})$ is generated by

$$x_{\alpha_1}(f) = \begin{pmatrix} 1 & f & \\ & 1 & \\ & & 1 \end{pmatrix} \quad x_{\alpha_2}(f) = \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \quad x_\varphi(f) = \begin{pmatrix} 1 & & f \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$x_{-\alpha_1}(f) = \begin{pmatrix} 1 & & \\ f & 1 & \\ & & 1 \end{pmatrix} \quad x_{\alpha_2}(f) = \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \quad x_\varphi(f) = \begin{pmatrix} 1 & & \\ & 1 & \\ f & & 1 \end{pmatrix}$$

$$h_{\alpha_1^\vee}(g) = \begin{pmatrix} g & & \\ & g^{-1} & \\ & & 1 \end{pmatrix} \quad \text{and} \quad h_{\alpha_2^\vee}(g) = \begin{pmatrix} 1 & & \\ & g & \\ & & g^{-1} \end{pmatrix}$$

for $f \in \mathbb{F}$ and $g \in \mathbb{F}^\times$.

The flag variety (= building) G/B

Let
$$B = \langle x_\alpha(f), h_{\lambda^\vee}(g) \mid \alpha \in R^+, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}, f \in \mathbb{F}, g \in \mathbb{F}^\times \rangle$$

Theorem [Steinberg, Thm 15 and Lemma 43]

(b) Let $w \in W_0$ and $w = s_{i_1} \cdots s_{i_\ell}$ a reduced word.

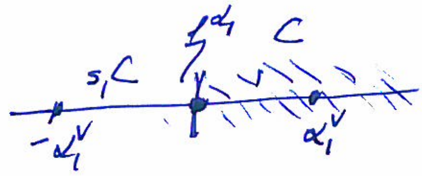
$\{ x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell) n_{i_\ell}^{-1} \}$ is a set of coset representatives of B cosets in BwB.

(a) Let $x_i(c) = x_{\alpha_i}(c)$ and $n_i = n_{\alpha_i}(1)$.
$$G = \bigsqcup_{w \in W_0} BwB \quad , \quad \text{where}$$

$BwB = Bn_wB$, $n_w = n_{i_1}^{-1} \cdots n_{i_\ell}^{-1}$ and $n_i = n_{\alpha_i}(1)$.

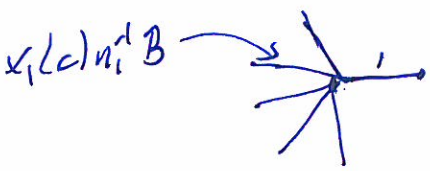
Example $SL_2(\mathbb{F}_p)/B$.

$W_0 = S_2$ acting on $\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\alpha_1^\vee$



$SL_2(\mathbb{F}_p)/B = B \sqcup Bs_1B$ and

$Bs_1B = Bn_1^{-1}B = \{ x_1(c) n_1^{-1} B \mid c \in \mathbb{F}_p \}$



Loop groups

$\mathbb{Q}_p = \{ \sum_{j=-\ell}^{\infty} a_j p^j \mid \ell \in \mathbb{Z},\ a_j \in \{0,1,\ldots,p-1\} \}$ is the field of fractions of

$\mathbb{Z}_p = \{ \sum_{j=0}^{\infty} a_j p^j \mid a_j \in \{0,1,\ldots,p-1\} \}$.

$$G = G(\mathbb{Q}_p)$$
$$\cup|$$

stabilizer of a vertex — maximal compact subgroup $K = G(\mathbb{Z}_p) \xrightarrow[\Phi]{p=0} G(\mathbb{F}_p)$

$$\cup| \qquad\qquad\qquad \cup|$$

stabilizer of an alcove — Iwahori subgroup $I = \Phi^{-1}(B(\mathbb{F}_p)) \longrightarrow B(\mathbb{F}_p)$

G/I = affine flag variety = "affine building".

G/K = loop Grassmannian

Let $t_{\lambda^\vee} = h_{\lambda^\vee}(t^{-1})$. The <u>affine Weyl group</u> is

$$W = W_0 \ltimes \mathfrak{h}_{\mathbb{Z}} = \{ w t_{\lambda^\vee} \mid w \in W_0,\ \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}} \}$$

Then

$$G = \bigsqcup_{w \in W} IwI \quad \text{and} \quad G = \bigsqcup_{v \in W} U^- v I$$

$$G = \bigsqcup_{\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}/W_0} K t_{\lambda^\vee} K \quad \text{and} \quad G = \bigsqcup_{\mu^\vee \in \mathfrak{h}_{\mathbb{Z}}} U^- t_{\mu^\vee} K$$

where $U^- = \langle x_{-\alpha}(f) \mid f \in \mathbb{Q}_p,\ \alpha \in R^+ \rangle$.

The <u>Mirković-Vilonen intersections</u> are

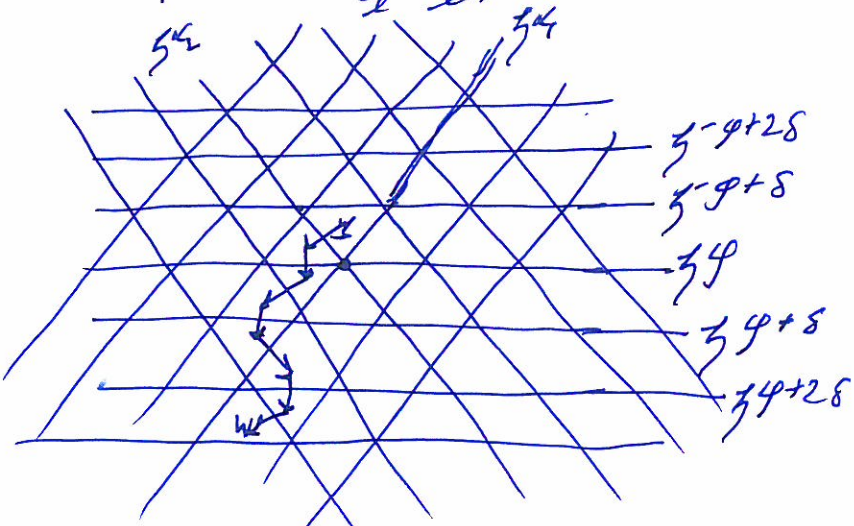
$$IwI \cap U^- v I \quad \text{and} \quad K t_{\lambda^\vee} K \cap U^- t_{\mu^\vee} K.$$

Theorem ~~[illegible]~~

$$\left\{\begin{array}{c}\text{labeled positively folded}\\ \text{paths of type } \vec{w} \text{ and end } v\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{coset representatives of}\\ I\text{-cosets in}\\ IwI \cap U^- vI\end{array}\right\}$$

$$p = (p_1, p_2, \ldots, p_\ell) \longmapsto x_{\gamma_1}(d_1) \cdots x_{\gamma_\ell}(d_\ell) v$$

First note

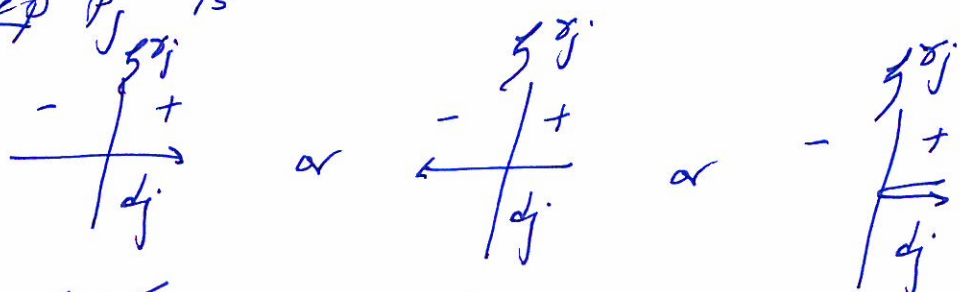


$W \leftrightarrow \{\text{alcoves}\}$

$x_{\alpha+k\delta}(c) = x_\alpha(cp^k)$

Let $w \in W$ and $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ a path to w, minimal length

A step p_j is



$d_j \in \mathbb{F}_p$ $\qquad$ $d_j = 0$ $\qquad$ $d_j \in \mathbb{F}_p^\times$.

where the periodic orientation has

(a) hyperplanes through 0 have 1 on the positive side

(b) parallel hyperplanes have parallel orientation.

Chevalley groups

Let $\mathbb{F}$ be a field (or commutative ring)

The Chevalley group $G(\mathbb{F})$ corresponding to $(W_0, \mathfrak{h}_{\mathbb{Z}})$ and $\mathbb{F}$ is given by generators

$$x_\alpha(f), \quad x_{-\alpha}(f), \quad h_{\lambda^\vee}(g)$$

$\alpha \in R^+$, $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$, $f \in \mathbb{F}$, $g \in \mathbb{F}^\times$ with

$$x_{\pm\alpha}(f_1)x_{\pm\alpha}(f_2) = x_{\pm\alpha}(f_1+f_2) \qquad \boxed{n_\alpha(g)x_\alpha(f)n_\alpha(g)^{-1} = x_{-\alpha}(-g^{-2}f)} \quad \times$$

$$\boxed{h_{\alpha^\vee}(g_1)h_{\alpha^\vee}(g_2) = h_{\alpha^\vee}(g_1g_2)} \qquad h_{\alpha^\vee}(g) = n_\alpha(g)n_\alpha(1)^{-1}$$

$$h_{\lambda^\vee}(g_1)h_{\lambda^\vee}(g_2) = h_{\lambda^\vee}(g_1g_2) \quad \text{and} \quad h_{\lambda^\vee}(g)h_{\mu^\vee}(g) = h_{\lambda^\vee+\mu^\vee}(g)$$

$$n_\alpha(g)x_\beta(f)n_\alpha(g)^{-1} = x_{s_\alpha\beta}(\epsilon_{\alpha\beta}\, g^{-\langle\beta,\alpha^\vee\rangle}f)$$

$$h_{\lambda^\vee}(g)x_\beta(f)h_{\lambda^\vee}(g)^{-1} = x_\beta(g^{\langle\beta,\lambda^\vee\rangle}f)$$

$$n_\alpha(g)h_{\lambda^\vee}(g')n_\alpha(g)^{-1} = h_{s_\alpha\lambda^\vee}(g').$$

where $n_\alpha(g) = x_\alpha(g)x_{-\alpha}(-g^{-1})x_\alpha(g)$.

Hence

- G has a symmetry under the subgroup

  N generated by T and $n_{\pm\alpha}(g)$, $\alpha \in R^+$, $g \in \mathbb{F}^\times$.

- G contains one copy of $SL_2(\mathbb{F})$ for each $\alpha \in R^+$,

  $$SL_2(\mathbb{F}) \simeq \langle x_\alpha(f), x_{-\alpha}(f) \mid f \in \mathbb{F} \rangle.$$

SCHEDULE

	Wednesday	Thursday	Friday
9:00am - 9:10am	Welcome	–	–
9:10am - 10:00am	George Willis	Arun Ram	Martin Liebeck
10:00am - 10:30am	Brian Alspach	Jacqui Ramagge	Cai-heng Li
10:30am - 11:00am	Morning tea	Morning tea	Morning tea
11:00am - 11:30am	Murray Elder	Anthony Henderson	Anne Thomas
11:30am - 12:00pm	David Pask	Alice Devillers	Michael Whittaker
12:00pm - 1:40pm	Lunch	Lunch	Lunch
1:40pm - 2:30pm	Ákos Seress	Cheryl Praeger	Brendan McKay
2:30pm - 3:00pm	Discussion Session I	Discussion Session II	Discussion Session III
3:00pm - 3:30pm	Afternoon tea	Afternoon tea	Afternoon tea
3:30pm - 4:00pm	James Parkinson	John Bamberg	Discussion Session IV
4:00pm - 4:30pm	Michael Giudici	Lawrence Reeves	