Elliptic Schubert Calculus Colloquium UCLA 26.04.2012

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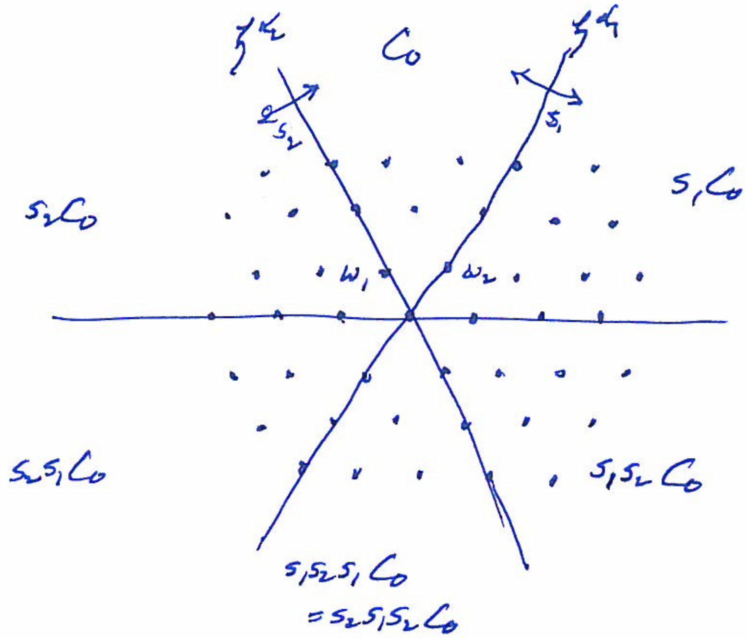
Picture of $SL_3(\mathbb{C}) = \{g \in M_{3\times 3}(\mathbb{C}) \mid \det(g) = 1\}$



$W_0 = \langle s_1, s_2 \mid s_i^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle = N(T)/T$

$\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = Hom(T, \mathbb{C}^\times)$

$\mathfrak{h}_{\mathbb{Z}} = Hom(\mathbb{C}^\times, T)$.

$$\begin{array}{l} G(\mathbb{C}) = SL_3(\mathbb{C}) \\ \ \ \cup \qquad\quad \cup \\ B = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ 0 & & * \end{pmatrix} \right\} \quad \text{Borel subgroup.} \\ \cup \\ T = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad \text{maximal torus.} \end{array}$$

$$P_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \qquad P_2 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$$

← parabolic subgroups

Partial flag varieties

$$\begin{tikzcd} & G/B \arrow[dl] \arrow[dr] & \\ G/P_2 \arrow[dr] & & G/P_1 \arrow[dl] \\ & G/G & \end{tikzcd}$$

Schubert varieties

$$\begin{tikzcd} & X_{s_1 s_2 s_1} & \\ X_{s_2 s_1} \arrow[ur, hook] & & X_{s_1 s_2} \arrow[ul, hook] \\ X_{s_2} \arrow[u, hook] \arrow[urr, hook] & & X_{s_1} \arrow[u, hook] \arrow[ull, hook] \\ & X_1 \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

T-fixed points

$z_w : pt \to G/B$
$* \mapsto wB$.

The Bruhat decomposition

$$G = \bigsqcup_{w \in W_0} BwB \quad \text{and} \quad X_w = \overline{BwB} \ \text{in } G/B$$

<u>Generalised cohomology theories</u> (Whitehead/Adams/May ...)

(1) $H_T(G/B)$ equivariant cohomology
(..., Bernstein-Gelfand-Gelfand, ...)

(2) $K_T(G/B)$ equivariant K-theory
(... Demazure, Kostant-Kumar, ...)

(3) $Ell_T(G/B)$ equivariant elliptic cohomology
(... Grojnowski, Ginzburg-Kapranov-Vasserot, Ando ...)

(4) $\Omega_T(G/B)$ equivariant cobordism
(... Bressler-Evens, Hornbostel-Kiritchenko, Calmès-Petrov-Zainoulli

<u>Axioms/Tools</u> (0) Normalization $H_T(pt)$, $K_T(pt)$, ...

(1) Products/smashes/suspensions/cofibrations $H_{G\times K}(M\times N)$

(2) Functoriality: If $f: X\to Y$ then $f^*: H_T(Y)\to H_T(X)$

(3) Thom isomorphism/orientability: If $f: X\to Y$ then
try to make $f_*: H_T(X)\to H_T(Y)$

(4) Change of groups: If $\varphi: G\to K$ then try to make
$\mathcal{X}^\varphi: H_K\to H_G$ and $\mathcal{X}_\varphi: H_G\to H_K$.

Representations: Hermann Weyl

$\mathring{\mathfrak{h}}^*_{\mathbb{Z}}$ indexes irreducible representations $X^\mu : T \to \mathbb{C}^\times$

If M is a G-module (vector space with G-action)

$$M = \bigoplus_{\mu \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}} M_\mu \quad \text{with} \quad M_\mu = \{ m \in M \mid tm = X^\mu(t) m \text{ for } t \in T \}.$$

Define

$$\mathrm{char}(M) = \sum_{\mu \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}} \dim(M_\mu) e^\mu$$

in $\mathbb{C}[\mathring{\mathfrak{h}}^*_{\mathbb{Z}}] = \mathrm{span}\{ e^\lambda \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}} \}$ with $e^\lambda e^\mu = e^{\lambda+\mu}$.

The irreducible G-modules (no submodules) have

$$\mathrm{char}(L(\lambda)) = \sum_{w \in W_0} \det(w) e^{w(\lambda+\rho)} \Big/ \sum_{w \in W_0} \det(w) e^{w\rho}.$$

Representations of $G(\mathbb{C}((t)))$: Kac-Peterson.

$$\mathbb{C}((t)) = \{ a_{-\ell} t^{-\ell} + a_{-\ell+1} t^{-\ell+1} + \cdots \mid \ell \in \mathbb{Z},\ a_i \in \mathbb{C} \}.$$

The affine Weyl group

$$W = W_0 \ltimes \mathring{\mathfrak{h}}_{\mathbb{Z}} = \{ w t_\beta \mid w \in W_0, \beta \in \mathring{\mathfrak{h}}_{\mathbb{Z}} \} \text{ with } w t_\beta = t_{w\beta} w,\ t_\beta t_\gamma = t_{\beta+\gamma}.$$

acts on

$$\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\delta \oplus \mathring{\mathfrak{h}}_{\mathbb{Z}} \oplus \mathbb{Z}\Lambda_0 \quad \text{and}$$

$$\Theta_{a\delta+\lambda+m\Lambda_0} = e^{\frac{-1}{2m}\|\lambda\|^2\delta} \sum_{\beta \in \mathring{\mathfrak{h}}_{\mathbb{Z}}} e^{t_\beta(a\delta+\lambda+m\Lambda_0)} = e^{m\Lambda_0} \sum_{\beta \in \mathring{\mathfrak{h}}_{\mathbb{Z}}} e^{\lambda+m\beta} q^{\frac{1}{2m}\|\lambda+m\beta\|^2}, \text{ with } q = e^{-\delta}.$$

The irreducible integrable $G(\mathbb{C}((t)))$-modules $L(a\delta+\bar{\lambda}+m\Lambda_0)$ have

$$\mathrm{char}\, L(a\delta+\lambda+m\Lambda_0) = \frac{\sum_{w\in W_0} \det(w)\,\Theta_{w(a\delta+\bar{\lambda}+m\Lambda_0+\rho+g\Lambda_0)}}{\sum_{w\in W_0} \det(w)\,\Theta_{w(\rho+g\Lambda_0)}}.$$

4 rings/4 cohomology theories

(1) $H_T(pt) = S(\mathfrak{h}^{\circ *}_{\mathbb{Z}}) = \mathbb{C}[x_1,\ldots,x_n]$ if $\mathfrak{h}^{\circ *}_{\mathbb{Z}}$ has basis $x_1,\ldots,x_n$.

$H_G(pt) = H_T(pt)^{W_0} = \mathbb{C}[x_1,\ldots,x_n]^{W_0}$

$$H_T(G/B) = H_T(pt)\otimes_{H_G(pt)} H_T(pt) = \frac{\mathbb{C}[y_1,\ldots,y_n,x_1,\ldots,x_n]}{\langle f(x_1,\ldots,x_n)-f(y_1,\ldots,y_n) \mid f\in\mathbb{C}[x_1,\ldots,x_n]^{W_0}\rangle}$$

(2) $K_T(pt) = \mathbb{C}[\mathfrak{h}^{\circ *}_{\mathbb{Z}}] = \mathrm{span}\{e^{\lambda} \mid \lambda\in\mathfrak{h}^{\circ *}_{\mathbb{Z}}\}$ with $e^{\lambda}e^{\mu}=e^{\lambda+\mu}$

$= \mathbb{C}[X_1^{\pm 1},\ldots,X_n^{\pm 1}]$ with $X_i = e^{x_i}$.

$K_G(pt) = \mathbb{C}[\mathfrak{h}^{\circ *}_{\mathbb{Z}}]^{W_0} = \mathbb{C}[X_1^{\pm 1},\ldots,X_n^{\pm 1}]^{W_0}$

$K_T(G/B) = K_T(pt)\otimes_{K_G(pt)} K_T(pt)$

$$= \frac{\mathbb{C}[X_1^{\pm 1},\ldots,X_n^{\pm 1},Y_1^{\pm 1},\ldots,Y_n^{\pm 1}]}{\langle f(X_1,\ldots,X_n)-f(Y_1,\ldots,Y_n) \mid f\in\mathbb{C}[X_1^{\pm 1},\ldots,X_n^{\pm 1}]^{W_0}\rangle}$$

(3) $Ell_T(pt)$ depends on

$$E_\tau = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}} \text{ an elliptic curve } \left(\begin{array}{l}\tau\in\mathbb{C},\ \mathrm{Im}(\tau)>0 \\ E_\tau \text{ a projective variety}\end{array}\right).$$

and is a sheaf on the abelian variety

$$E_\tau^{\mathfrak{h}} = \frac{\mathfrak{h}_{\mathbb{C}}^{0*}}{\mathfrak{h}_{\mathbb{Z}}^{0*}+\tau\mathfrak{h}_{\mathbb{Z}}^{0*}} \qquad \text{where } \mathfrak{h}_{\mathbb{C}}^{0*} = \mathbb{C}\otimes_{\mathbb{Z}}\mathfrak{h}_{\mathbb{Z}}^{0*}.$$

The ~~non~~degenerate form $\langle\,,\,\rangle$ on $\mathfrak{h}_{\mathbb{Z}}^{0*}$ provides an ample line bundle $\mathcal{L}$ and

$$\tau\widetilde{Th} = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} H^0(E_\tau^{\mathfrak{h}}, \mathcal{L}^{\otimes m})$$

is $\widetilde{Th}$ the ring generated by $\Theta_{a\delta+\lambda+m\Lambda_0}$ at $q=e^{2\pi i\tau}$. So

$Ell_T(pt)$ "=" $\widetilde{Th}$,

$Ell_G(pt)$ "=" $\widetilde{Th}^{W_0}$,

and $Ell_T(G/B)$ "=" $\widetilde{Th}\otimes_{\widetilde{Th}^{W_0}}\widetilde{Th}$.

is combinatorially tractable with a complete analogue of the combinatorial theory of Schubert polynomials.