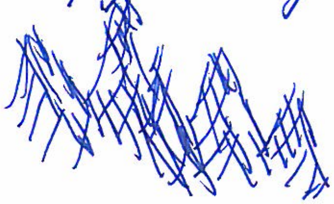
Views from Castalia Colloquium USC 27.04.2012

Preprojective algebra Λ



Λ_b = irreducible components

Quiver variety/KLR algebras R

L_b simple modules

G/K the loop Grassmannian

Z_b = MV cycles

char

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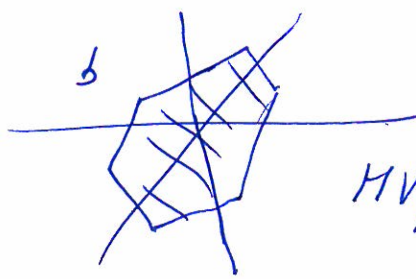
$\mathrm{char}(\Lambda_b)$ semicanonical basis

$\mathrm{char}(L_b)$ canonical basis

$\mathrm{char}(Z_b)$ MV basis

in $U_q n^-$ the quantum group

b



MV polytope

The crystal structure in all three cases is exactly the same.

G/K and Dynkin diagrams

$$\mathbb{C}((t)) = \{a_{-\ell}t^{-\ell} + a_{-\ell+1}t^{-\ell+1} + \cdots \mid a_i \in \mathbb{C},\ \ell \in \mathbb{Z}\}$$
$$\cup\!|$$
$$\mathbb{C}[[t]] = \{a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i \in \mathbb{C}\} \xrightarrow{t=0} \mathbb{C}$$

G a complex reductive algebraic group, $SL_3(\mathbb{C})$, $GL_n(\mathbb{C})$ or $Sp_4(\mathbb{C}$

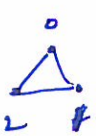
$$G = SL_3(\mathbb{C}((t)))$$
$$\cup\!|$$
$$K = SL_3(\mathbb{C}[[t]]) \xrightarrow{t=0} SL_3(\mathbb{C})$$

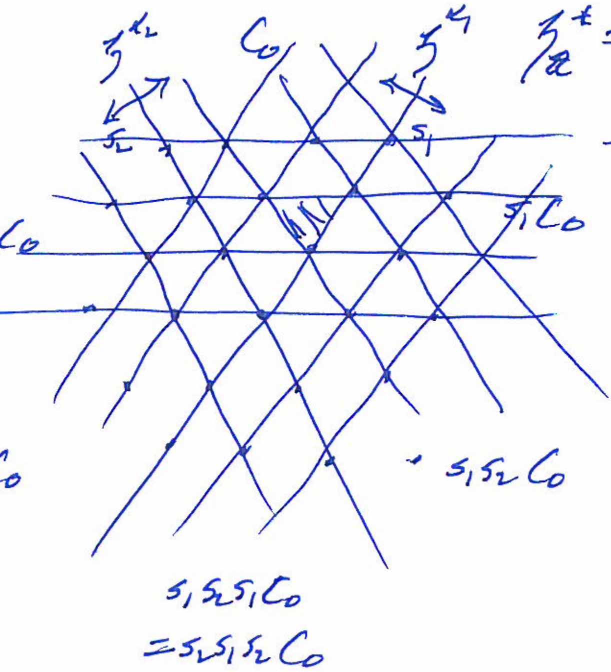
Then $T = \left\{\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}\right\}$ is a maximal torus of $SL_3(\mathbb{C})$

The <u>Weyl group</u> and <u>character lattice</u> are

$$W_0 = N(T)/T \quad \text{and} \quad \mathfrak{h}^*_{\mathbb{Z}} = \mathrm{Hom}(T, \mathbb{C}^\times).$$

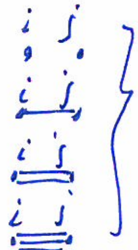
<u>For SL_3:</u> $W_0 = \langle s_1, s_2 \mid s_i^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$





$\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.

The extended or affine Dynkin diagram is the dual graph of the fundamental alcove with

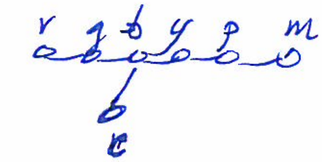


if $\mathfrak{h}^{\alpha_i} \sphericalangle \mathfrak{h}^{\alpha_j}$ is $\begin{cases} \pi/2 \\ \pi/3 \\ \pi/4 \\ \pi/6 \end{cases}$

Quiver Hecke algebras / KLR algebras R_d, $d \in \mathbb{Z}_{\geq 0}$

The set of colors is

$$I = \{\text{labels of the Dynkin diagram}\}$$



R_d is generated by

$$y_1, \ldots, y_d, \quad e_u \text{ for } u \in I^d, \quad \psi_1, \ldots, \psi_{d-1}$$

with relations

$$y_i y_j = y_j y_i, \quad e_u e_v = \delta_{uv} e_u, \quad \psi_i e_u = e_{s_i u} \psi_i$$

and more

where $I^d = \{u = (u_1, \ldots, u_d) \text{ sequences of length } d \text{ in } I\}$

$s_i u$ is u except i^{th} and $(i+1)^{\text{st}}$ are switched.

and $\mathbb{Z}$-grading

$$\deg(y_i) = 2, \quad \deg(e_u) = 0, \quad \deg(\psi_i e_u) = \begin{cases} -2 & \text{if } u_i = u_{i+1} \\ 1 & \text{if } u_i \quad u_{i+1} \\ 0 & \text{if } u_i \quad u_{i+1}. \end{cases}$$

If M is a $\mathbb{Z}$-graded R-module

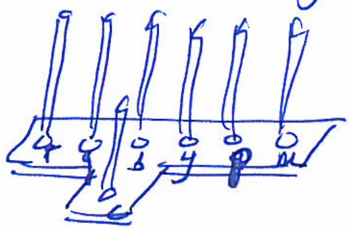
$$\mathrm{char}(M) = \sum_{j \in \mathbb{Z}} \sum_{u \in I^d} \dim(e_u M[j])\, q^j f_{u_1} \cdots f_{u_d}$$

Theorem (Khovanov-Lauda/Rouquier)

$$\mathrm{char}: \begin{array}{c}\text{Grothendieck} \\ \text{group of}\end{array} \left\{\begin{array}{c}\text{fin. dim. } \mathbb{Z}\text{graded} \\ R\text{-modules}\end{array}\right\} \longrightarrow U_q\mathfrak{n}^- \text{ the quantum group}$$

simple modules $\quad L_b \longmapsto \mathrm{char}(L_b)$ canonical basis

The Glass Bead game

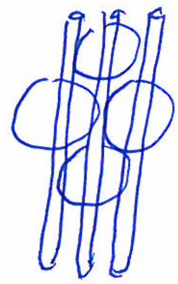


Board

Beads

A *skew shape* is a configuration of beads λ such that any two beads on the same runner are separated by at least two beads, i.e.

or



A *standard tableau* of shape λ is a sequence of runners $T = (T_1, \ldots, T_d)$ which results in λ when played.

Define

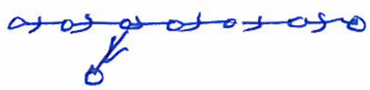
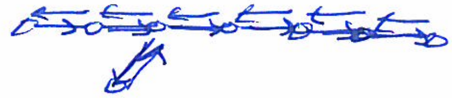
$$L_\lambda = \mathrm{span}\{ v_T \mid T \text{ is a standard tableau of shape } \lambda \}$$

$$y_i v_T = 0, \qquad e_u v_T = \delta_{u_T} v_T, \qquad \psi_i v_T = \begin{cases} v_{s_i T}, & \text{if } s_i T \text{ is standard of shape } \lambda \\ 0, & \text{otherwise.} \end{cases}$$

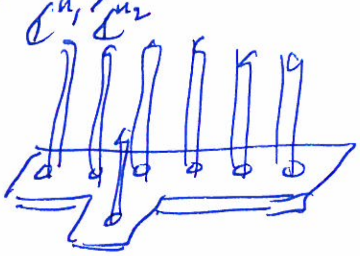
Theorem [Kleshchev-Ram]

L_λ is a simple R-module.

Quiver / Preprojective variety

 and 

Idea Replace beads by vector spaces



$\mathbb{C}^{n_j}$ corresponds to n_j beads on runner j.

The data: a vector space on each runner
a linear transformation for each edge
is a representation of the quiver.

$$\Lambda = \prod_{\text{edges}} M_{n_i n_j}(\mathbb{C}) \Big/ \sim$$

where $\sim$ is equivalence of representations (change of bases in vector spaces).

Λ_λ are irreducible components.

Philosophy

Λ = preprojective variety = $T^*(X)$ | X = quiver variety | G/K loop Grassmanian

n^- = Lie algebra | $U_q n^-$ = quantum group | G the Lie group.

orbits/functions on Λ $\rightsquigarrow$ sheaves/bundles functions on X | orbits/functions on G/K

characteristic variety

exponential map.