On T. Church, J. Ellenberg and B. Farb, FI-modules: a new approach to stability for S_n-representations, 22.06.2012 ①

Univ. Melbourne, Seminar
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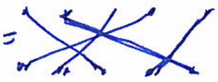
Notation for S_n

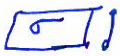
(1) The symmetric group S_n is

$$S_n = \{\sigma : \{1,\dots,n\} \to \{1,\dots,n\} \mid \sigma \text{ is a bijection}.$$

Identify σ and $\sigma =$  with i in top row connected to $\sigma(i)$ in bottom row.

Let

$$\iota : S_n \hookrightarrow S_{n+1}$$

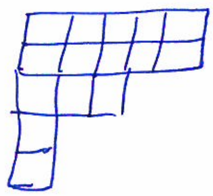
$\sigma \longmapsto$ 

(2) The irreducible S_n-modules S_n^λ are indexed by $\lambda \in \hat{S}_n$, the set of partitions with n boxes.

$$\hat{S}_n = \left\{ (\lambda_1,\dots,\lambda_\ell) \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0},\ \lambda_i \in \mathbb{Z},\ \lambda_1 \geq \cdots \geq \lambda_\ell \\ \lambda_1 \geq \cdots \geq \lambda_\ell > 0,\ \lambda_1 + \cdots + \lambda_\ell = n \end{array} \right\}$$

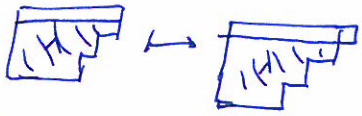
$$= \left\{ (\lambda_2, \lambda_3, \dots \lambda_\ell) \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0},\ \lambda_i \in \mathbb{Z}, \\ n - (\lambda_2 + \cdots + \lambda_\ell) \geq \lambda_2 \geq \cdots \geq \lambda_\ell \end{array} \right\}$$

Identify

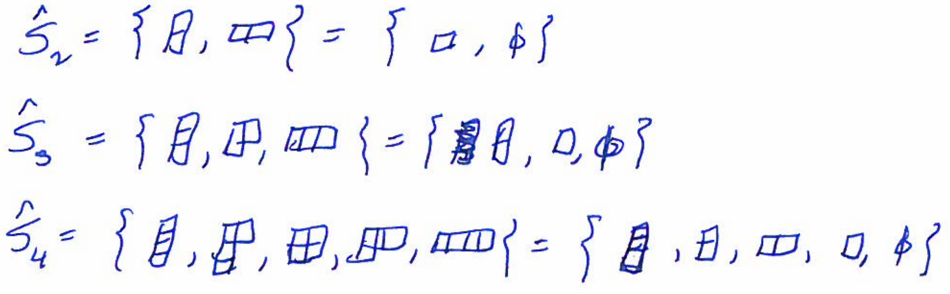
λ with  λ_i boxes in row i.

Let

$$\iota : \hat{S}_n \longrightarrow \hat{S}_{n+1}$$



For example



Representation Stability

Let
$$V = (V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} \cdots) \quad \text{with}$$

V_n an S_n-module and

$\phi_n : V_n \to V_{n+1}$ an S_n-module homomorphism,

where V_{n+1} is an S_n-module via $\iota : S_n \hookrightarrow S_{n+1}$.

The sequence V is <u>representation stable</u>

(a) There exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N$ then

(a1) $\phi_n : V_n \to V_{n+1}$ is injective and

(a2) $V_{n+1} = (\mathbb{C}S_{n+1})\text{-span}(\mathrm{im}\,\phi_n)$.

(b) If $\lambda \in \hat{S}$ then there exists $N_\lambda \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N_\lambda$ then $c_{\lambda,n} = c_{\lambda,N_\lambda}$.

The sequence V is <u>uniformly representation stable</u> if V satisfies (a) and

(B) There exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N$ and $\lambda \in \hat{S}_n$ then

$$c_{\lambda,n} = \begin{cases} c_{\lambda,N}, & \text{if } \lambda \in \hat{S}_N, \\ 0, & \text{if } \lambda \notin \hat{S}_n. \end{cases}$$

~~Some~~ categories

$\mathcal{F}$ is the category with

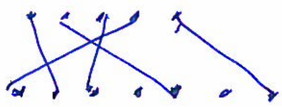
Objects: $\{1,2,\dots,m\}$, $m\in\mathbb{Z}_{\geq 0}$

Morphisms: $\mathrm{Hom}(\{1,\dots,m\},\{1,\dots,n\}) = \{\sigma : \{1,\dots,m\}\to\{1,\dots,n\} \mid \sigma \text{ is bijective}\}$.

FI is the category with

Objects: $\{1,2,\dots,m\}$, $m\in\mathbb{Z}_{\geq 0}$

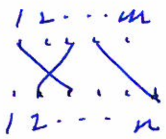
Morphism: $\mathrm{Hom}(\{1,\dots,m\},\{1,\dots,n\}) = \{\sigma : \{1,\dots,m\}\to\{1,\dots,n\} \mid \sigma \text{ is injective}\}$.

Identify σ with  where i in the top row is connected to $\sigma(i)$ in the bottom row.

~~Remark~~

FI# is the category with

Objects: $\{1,\dots,m\}$, $m\in\mathbb{Z}_{\geq 0}$

Morphisms: $\mathrm{Hom}(\{1,\dots,m\},\{1,\dots,n\}) = \Big\{$ 

singleton dot allowed on both top and bottom row, otherwise injective $\Big\}$

Remark: This is closely related to the rook monoid ($m\times n$ rook nonattacking placements).

An FI-module is a functor

$$V : FI \longrightarrow Vect$$
$$\{1,\dots,n\} \longmapsto V_n$$

$\left(\text{Think: } V = \bigoplus_n V_n\right)$

An FI module V is finitely generated if there exists $k\in\mathbb{Z}_{>0}$ and $v_1,\dots,v_k\in V$ such that

the minimal FI-submodule containing $v_1,\dots,v_k$ is V.

<u>Theorem</u> Let V be an FI-module.

(a) V is finitely generated if and only if V is uniformly representation stable and $\dim(V_n) < \infty$

(b) If V is finitely generated then there exists $P \in \mathbb{Q}[t]$ and $N \in \mathbb{Z}_{\geq 0}$ such that if $n \in \mathbb{Z}_{\geq 0}$ and $n \geq N$ then $\dim V_n = P(n)$

<u>Examples</u>

(a) $H^i(M_{g,n}; \mathbb{Q})$ is a finitely generated FI-module.

(b) $H^i(\overline{M}_{g,n}; \mathbb{Q})$ is an FI-module: <u>not</u> finitely generated.

(c) $H^i(\mathrm{Conf}_n(M); \mathbb{Q})$ is a finitely generated FI-module.

where $M_{g,n}$ is the moduli space of genus g n pointed curves,

$\overline{M}_{g,n}$ is ~~the~~ Deligne-Mumford compactification of $M_{g,n}$

$\mathrm{Conf}_n(M) = \{(p_1, \ldots, p_n) \in M^n \mid p_i \neq p_j\}$.

(d) <u>Murnaghan's theorem</u>: $S_n^\lambda \otimes S_n^\mu = \bigoplus_\nu g_{\lambda\mu}^\nu S_n^\nu$

with $g_{\lambda\mu}^\nu$ independent of n for n large.

(e) For $\lambda \in \hat{S}_m$ define $M(\lambda)$ by $M(\lambda)_n = \mathrm{Ind}_{S_m \times S_{n-m}}^{S_n}(S_m^\lambda \otimes \mathrm{triv})$

For $m \in \mathbb{Z}_{\geq 0}$ define $M(m) = M(\mathbb{C}S_m)$.

(f) For $\lambda \in \hat{S}$ define $V(\lambda) = S^\lambda$ by $V(\lambda)_n = \begin{cases} S_n^\lambda, & \text{if } \lambda \in \hat{S}_n, \\ 0, & \text{if } \lambda \notin \hat{S}_n \end{cases}$

With this notation

$$H^2(\mathrm{Conf}_{\bullet}(\mathbb{R}^2);\mathbb{Q}) = M(\ydiagram{2,1}) \oplus M(\ydiagram{3,1})$$ so that

$$H^2(\mathrm{Conf}_n(\mathbb{R}^2)) = \left(S_n^{\ydiagram{1}} \oplus S_n^{\ydiagram{1,1}} \oplus S_n^{\ydiagram{2}} \oplus S_n^{\ydiagram{2,1}}\right) \oplus \left(S_n^{\ydiagram{1}} \oplus S_n^{\ydiagram{1,1}} \oplus S_n^{\ydiagram{2}} \oplus S_n^{\ydiagram{2,1}} \oplus S_n^{\ydiagram{3}} \oplus S_n^{\ydiagram{3,1}}\right)$$

$$= \left(S_n^{\ydiagram{1}}\right)^{\oplus 2} \oplus \left(S_n^{\ydiagram{1,1}}\right)^{\oplus 2} \oplus \left(S_n^{\ydiagram{2}}\right)^{\oplus 2} \oplus \left(S_n^{\ydiagram{2,1}}\right)^{\oplus 2} \oplus S_n^{\ydiagram{3}} \oplus S_n^{\ydiagram{3,1}}$$

for n large.