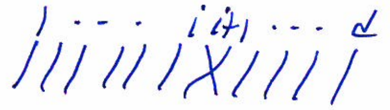
Lecture 2: Representations of the symmetric group
Brazil Algebra Conference, Salvador 17 July 2012

Representations of the symmetric group

The symmetric group is $S_d = \mathrm{Aut}(\{1,2,\ldots,d\})$.

Theorem The symmetric group is presented by generators $s_i =$  , $i = 1,2,\ldots,d-1$

with relations

(*) $s_i^2 = 1$ and $s_i s_j = s_j s_i$ if $j \neq i \pm 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

The degenerate affine Hecke algebra $\mathcal{H}_d$ is given by generators $x_1, \ldots, x_d$ and $s_1, \ldots, s_{d-1}$ with relations $x_i x_j = x_j x_i$ and (*) and

$s_i x_j = x_j s_i$ if $j \neq i, i+1$, $s_i x_i = x_{i+1} s_i + 1$,

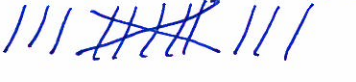
$s_i x_{i+1} = x_i s_i - 1$.

Theorem
The group algebra of S_d is $\mathcal{H}_d$ with additional relation $x_1 = 0$

i.e.

$$\begin{aligned} \mathcal{H}_d &\to \mathbb{C}S_d \\ s_i &\mapsto s_i \\ x_1 &\mapsto 0 \\ x_j &\mapsto \sum_{1 \le i < j} s_{ij} \end{aligned}$$

is an algebra homomorphism

where $s_{ij} =$ 

KLR Quiver Hecke algebras

I = vertex set of Dynkin diagram = {colours}.

The KLR quiver Hecke algebra R_d is given by generators

$$y_1, \dots, y_d, \quad e_u \text{ for } u \in I^d, \quad \psi_1, \dots, \psi_{d-1}$$

with relations

$$y_i y_j = y_j y_i, \quad e_u e_v = \delta_{uv} e_u, \quad 1 = \sum_{u \in I^d} e_u,$$

$$e_u y_i = y_i e_u, \quad e_u \psi_r = \psi_r e_{s_r u}, \quad \psi_r y_i = y_i \psi_r \text{ if } i \neq r, r+1,$$

$$\psi_r y_r e_u = \begin{cases} (y_{r+1}\psi_r + 1) e_u, & \text{if } (u_r, u_{r+1}) = (u_r, u_r) \\ y_{r+1}\psi_r e_u, & \text{otherwise.} \end{cases}$$

$$\psi_r y_{r+1} e_u = \begin{cases} (y_r \psi_r - 1) e_u, & \text{if } (u_r, u_{r+1}) = (u_r, u_r) \\ y_r \psi_r e_u, & \text{otherwise} \end{cases}$$

$$\psi_r \psi_s = \psi_s \psi_r, \text{ if } s \neq r, r\pm 1,$$

$$\psi_r^2 e_u = \begin{cases} 0, & \text{if } (u_r, u_{r+1}) = (u_r, u_r) \\ (y_{r+1} - y_r) e_u, & \text{if } (u_r, u_{r+1}) \text{ is } u_r \to u_{r+1} \\ -(y_{r+1} - y_r) e_u, & \text{if } (u_r, u_{r+1}) \text{ is } u_r \leftarrow u_{r+1} \\ e_u, & \text{otherwise} \end{cases}$$

$$\psi_r \psi_{r+1} \psi_r e_u = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1) e_u, & \text{if } (u_r, u_{r+1}, u_{r+2}) = (u_r, u_{r+1}, u_r) \\ & \quad \text{with } u_r \rightarrow u_{r+1} \\ (\psi_{r+1} \psi_r \psi_{r+1} - 1) e_u, & \text{if } (u_r, u_{r+1}, u_{r+2}) = (u_r, u_{r+1}, u_r) \\ & \quad \text{with } u_r \leftarrow u_{r+1} \\ \psi_{r+1} \psi_r \psi_{r+1} e_u, & \text{otherwise} \end{cases}$$

where $I^d = \{ u = (u_1, \ldots, u_d) \text{ sequences of length } d \text{ in } I \}$

$s_r u$ is u with u_r and u_{r+1} switched.

__Theorem__ If $Q = \cdots \underset{4}{\circ} - \underset{3}{\circ} - \underset{0}{\circ} - \underset{1}{\circ} - \underset{2}{\circ} \cdots$

then, after a completion,

$$\hat{\mathcal{H}}_d \cong \hat{R}_d$$

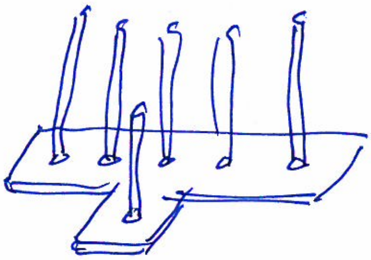
$\left(\mathbb{C}[[x_1, \ldots, x_d]] \text{ is a completion of } \mathbb{C}[x_1, \ldots, x_d] \text{ which contains } \frac{1}{1-x_1} = 1 + x_1 + x_1^2 + \cdots \right)$

__Theorem__ $\mathbb{C}S_d$ is R_d ~~with~~ for $Q = \cdots \underset{-1}{\circ} - \underset{0}{\circ} - \underset{1}{\circ} - \underset{2}{\circ} - \underset{3}{\circ} \cdots$

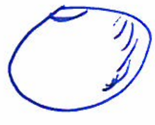
with

$e_u = 0$ if $u_1 \neq 0$ and $y_1 e_u = 0$ if $u_1 = 0$.

The Glass Bead game

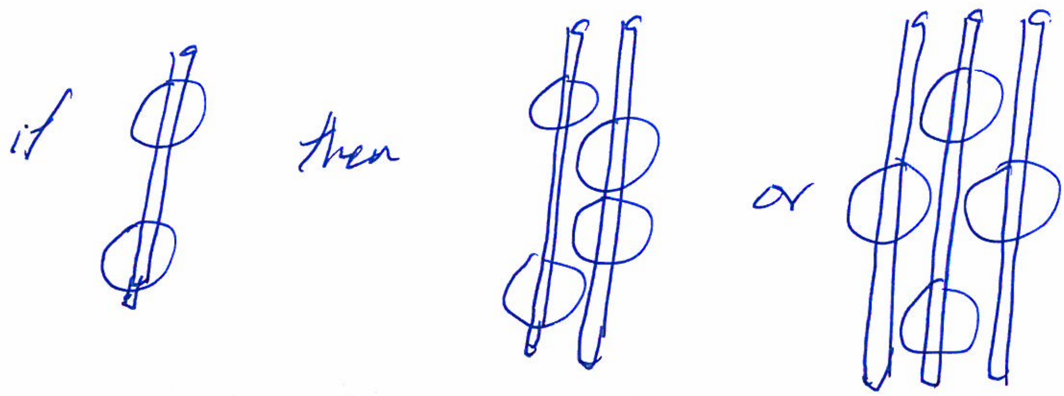


Board



Beads

A *skew shape* is a configuration of beads λ such that any two beads on the same runner are separated by two beads



A *standard tableau* of shape λ is a runner sequence $T=(T_1,\dots,T_d)$ which results in λ.

Define

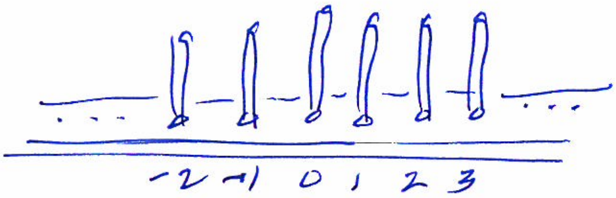
$$R_d^\lambda = \operatorname{span}\{ v_T \mid T \text{ is standard of shape } \lambda \}$$

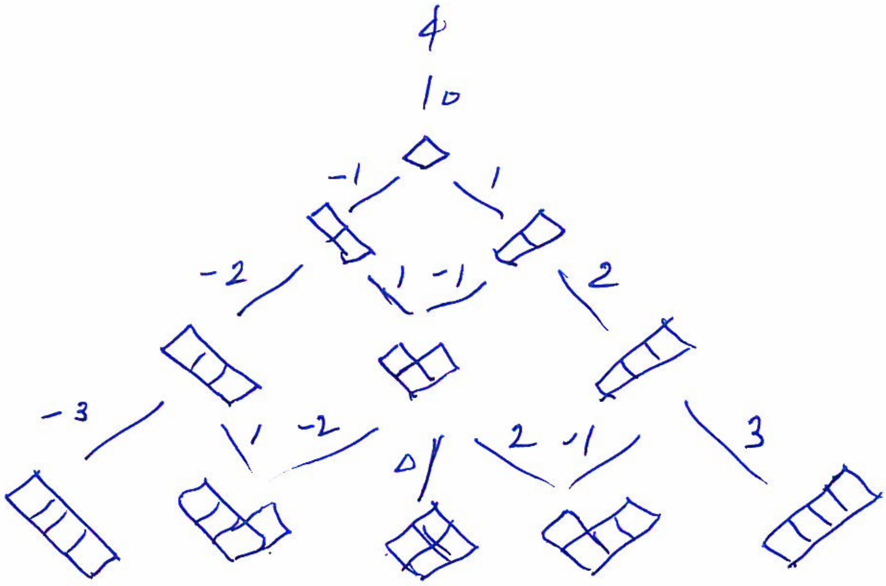
with

$$e_u v_T = \delta_{uT} v_T, \quad y_i v_T = 0, \quad \psi_r v_T = \begin{cases} v_{s_r T}, & \text{if } s_r T \text{ is standard shape} \\ 0, & \text{otherwise} \end{cases}$$

<u>Theorem</u> (Kleshchev–Ram) R_d^λ are simple R_d-modules.

<u>Young's lattice</u>





standard tableaux of shape λ correspond to paths from $\emptyset$ to λ.

<u>Theorem</u> R_d^λ for λ in Young's lattice (with d beads) are <u>all</u> simple $\mathbb{C}S_d$-modules.