XXII Brazilian Algebra Meeting, Salvador Bahia 18 July 2012.

Lecture 3: The Weyl Character formula

Symmetric functions

Initial data: $(W_0, \mathfrak{h}^*_{\mathbb{Z}})$ a finite $\mathbb{Z}$-reflection group

$\mathfrak{h}^*_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module

W_0 a finite subgroup of $GL(\mathfrak{h}^*_{\mathbb{Z}})$ generated by reflections

Example: Type GL_n $\quad \mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n\}$

$W_0 = S_n$ acting by permuting $\varepsilon_1, \ldots, \varepsilon_n$.

The group algebra of $\mathfrak{h}^*_{\mathbb{Z}}$ is

$$\mathbb{C}[X] = \text{span}\{X^\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\} \text{ with } X^\lambda X^\mu = X^{\lambda+\mu}.$$

W_0 acts on $\mathbb{C}[X]$ by $wX^\lambda = X^{w\lambda}$.

The ring of symmetric functions is

$$\mathbb{C}[X]^{W_0} = \{f \in \mathbb{C}[X] \mid wf = f \text{ for } w \in W_0\}.$$

Example: Type GL_3 Let $z_1 = X^{\varepsilon_1}$, $z_2 = X^{\varepsilon_2}$, $z_3 = X^{\varepsilon_3}$

$W_0 = S_3$, $\quad \mathbb{C}[X] = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$ and

$$\mathbb{C}[X]^{W_0} = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]^{S_3} = \mathbb{C}[e_1, e_2, e_3^{\pm 1}]$$

where

$$e_1 = z_1 + z_2 + z_3, \quad e_2 = z_1 z_2 + z_1 z_3 + z_2 z_3, \quad e_3 = z_1 z_2 z_3.$$

Weyl's Theorems

Let $G^\vee(\mathbb{C})$ be the reductive algebraic group corresponding to $(W_0, \mathfrak{h}_\mathbb{Z})$, $T^\vee$ a maximal torus of $G^\vee$.

(a) The simple $T^\vee$-modules X^λ are indexed by $\lambda \in \mathfrak{h}_\mathbb{Z}$

(b) The simple $G^\vee$-modules $L(\lambda)$ are indexed by $\lambda \in \mathfrak{h}_\mathbb{Z}^+$

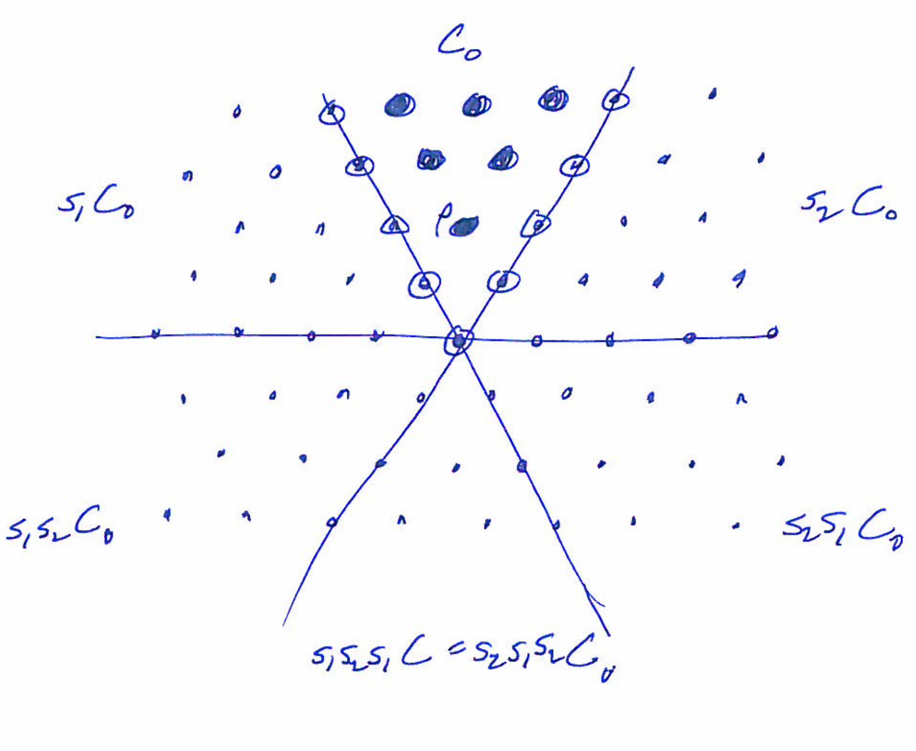
(c) The character of $L(\lambda)$ is

$$\mathrm{Res}^G_T(L(\lambda)) = s_\lambda$$

(d)
$$a_\rho = X^\rho \prod_{\alpha \in R^+} (1 - X^{-\alpha})$$

where R^+ is an index set for the reflections in W_0 such that

$$s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha^\# \quad \text{for } \alpha \in R^+,\ \mu \in \mathfrak{h}_\mathbb{Z}.$$



The affine Hecke algebra H

Let $\mathfrak{h}^{\alpha_1}, \ldots, \mathfrak{h}^{\alpha_\ell}$ be the walls of C_0,

$s_1, \ldots, s_\ell$ the corresponding reflections, so that

$$s_i : \mathfrak{h}_{\mathbb{Z}} \to \mathfrak{h}_{\mathbb{Z}} \quad \text{with} \quad s_i\lambda = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i .$$

The affine Hecke algebra H is generated by

$$T_1, \ldots, T_\ell \quad \text{and} \quad X^\lambda, \ \lambda \in \mathfrak{h}_{\mathbb{Z}}$$

with relations

$$\underbrace{T_iT_jT_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{T_jT_iT_j\cdots}_{m_{ij}\text{ factors}}, \quad \text{for } i \neq j \text{ with } \tfrac{\pi}{m_{ij}} = \mathfrak{h}^{\alpha_i} \measuredangle \mathfrak{h}^{\alpha_j} .$$

$$T_i^2 = (t^{\frac12} - t^{-\frac12})T_i + 1, \quad \text{for } i = 1, \ldots, \ell$$

$$X^\lambda X^\mu = X^{\lambda+\mu}, \quad \text{for } \lambda, \mu \in \mathfrak{h}_{\mathbb{Z}}$$

$$T_i X^\lambda = X^{s_i\lambda} T_i + (t^{\frac12} - t^{-\frac12}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}$$

Define

$$T_w = T_{i_1} \cdots T_{i_\ell} \quad \text{for a reduced word } w = s_{i_1} \cdots s_{i_\ell} .$$

Then

$$\{X^\lambda T_w \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}, w \in W_0\} \text{ is a basis of } H$$

and

$$\mathbb{C}[X] = \mathrm{span}\{X^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}\} \quad \text{and} \quad H_0 = \mathrm{span}\{T_w \mid w \in W_0\}$$

are subalgebras.

Geometric Langlands

Let $\mathbb{1}_0, \varepsilon_0 \in H_0$ be such that

$$\mathbb{1}_0^2 = \mathbb{1}_0 \quad \text{and} \quad T_i \mathbb{1}_0 = t^{\frac{1}{2}} \mathbb{1}_0 \qquad \text{for } i=1,\dots,\ell$$
$$\varepsilon_0^2 = \varepsilon_0 \qquad\qquad T_i \varepsilon_0 = (-t^{-\frac{1}{2}}) \varepsilon_0$$

Then

$$\begin{array}{ccc} H & \longrightarrow & H\mathbb{1}_0 = \mathbb{C}[X]\mathbb{1}_0 \\ h & \longmapsto & h\mathbb{1}_0 \end{array}$$

makes $\mathbb{C}[X]$ an H-module, the *polynomial representation* of H. Then

$$\begin{array}{ccccccc} \mathbb{C}[X]^{W_0} & = & Z(H) & \hookrightarrow & \mathbb{1}_0 H \mathbb{1}_0 & \hookrightarrow & \varepsilon_0 H \mathbb{1}_0 \\ f & \longmapsto & f & \longmapsto & f\mathbb{1}_0 & \longmapsto & A_\rho f \mathbb{1}_0 \\ s_\lambda & & \longleftarrow & & C_\lambda & \longleftarrow & A_{\lambda+\rho} \\ P_\lambda(0,t) & & \longleftarrow & & M_\lambda & & \end{array}$$

where $M_\lambda = \mathbb{1}_0 X^\lambda \mathbb{1}_0$ and $A_\mu = \varepsilon_0 X^\mu \mathbb{1}_0$,

C_λ is the *Kazhdan-Lusztig basis* of the *spherical Hecke algebra* $\mathbb{1}_0 H \mathbb{1}_0 = K(\mathrm{Perv}(G/K))$ the Grothendieck group of the category of perverse sheaves on the *loop Grassmannian* G/K.

$P_\lambda(0,t)$ is Macdonald's spherical function for $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$.