Lecture 4 Crystals from paths and MV polytopes

Brazil algebra conference

Crystals: The path model

Salvador 19 July 2012

Initial data: $(\mathfrak{h}_{\mathbb{Z}}, W_0)$ where

$\mathfrak{h}_{\mathbb{Z}}$ has $\mathbb{Z}$-basis

W_0 is a finite subgroup of $GL(\mathfrak{h}_{\mathbb{Z}})$ generated by reflections.

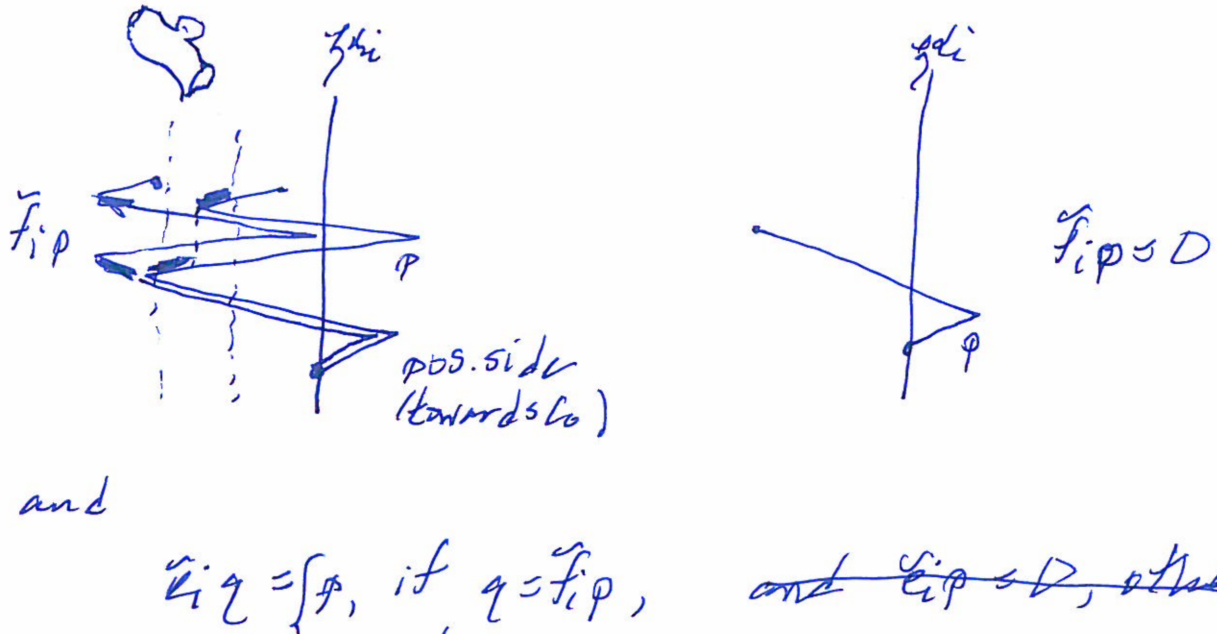
Let $\mathfrak{h}_{\mathbb{R}}$ = $\mathbb{R}$-span of the basis and

$$B_{univ} = \left\{ p : [0,1] \to \mathfrak{h}_{\mathbb{R}} \;\middle|\; \begin{array}{l} p \text{ is continuous, piecewise linear} \\ p(1) \in \mathfrak{h}_{\mathbb{Z}},\ p(0) = 0 \end{array} \right\}$$

A *crystal* is a subset B of B_{univ} closed under the root operators $\tilde{f}_1, \ldots, \tilde{f}_n, \tilde{e}_1, \ldots, \tilde{e}_n$.

C_0 is a fundamental chamber for W_0 acting on $\mathfrak{h}_{\mathbb{R}}$

$\mathfrak{h}^{\alpha_1}, \ldots, \mathfrak{h}^{\alpha_n}$ are the walls of C_0



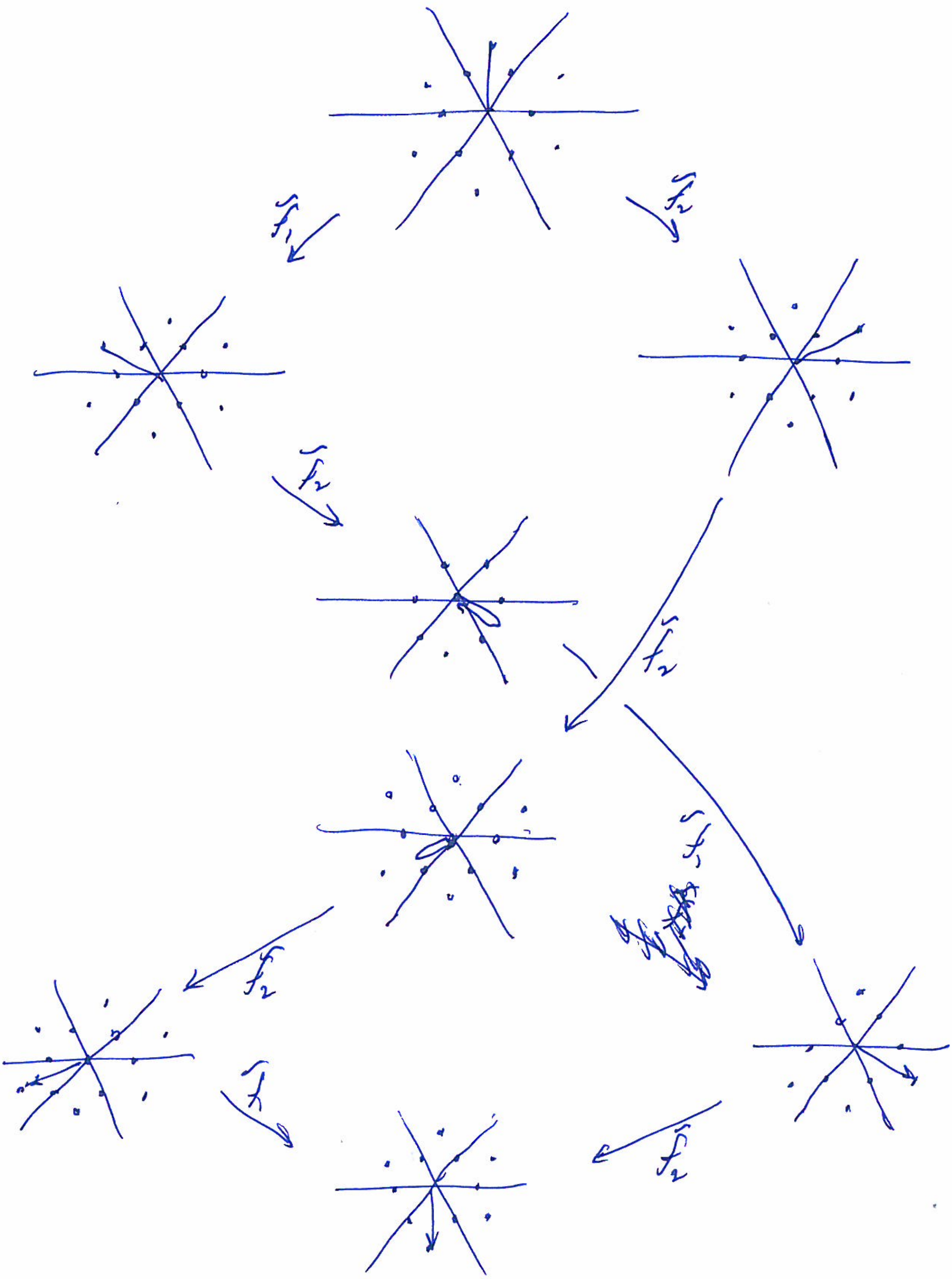
and

$$\tilde{e}_i q = \begin{cases} p, & \text{if } q = \tilde{f}_i p, \\ 0, & \text{otherwise,} \end{cases}$$

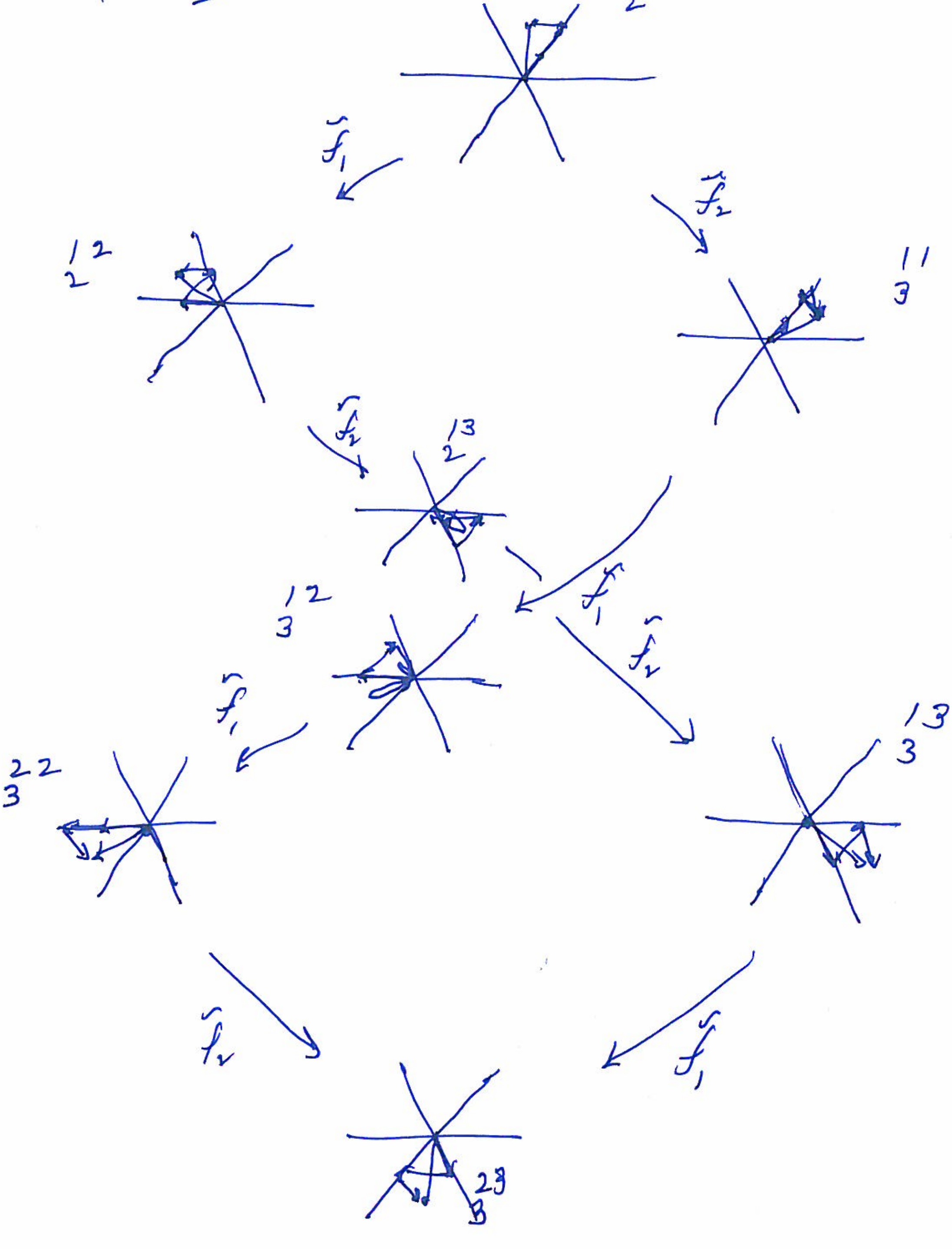
~~and $\tilde{e}_i p = 0$, otherwise.~~

$$\tilde{e}_i p = \begin{cases} q, & \text{if } \tilde{f}_i q = p \\ 0, & \text{otherwise.} \end{cases}$$

A crystal B is a subset of B_{univ} closed under the action of $\tilde{e}_1, \dots, \tilde{e}_n$ and $\tilde{f}_1, \dots, \tilde{f}_n$.

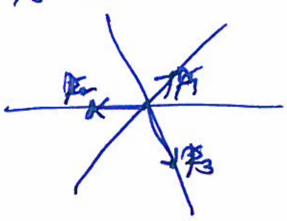


Example Type SL_3, $\lambda = \omega_1 + \omega_2$ (1.6)



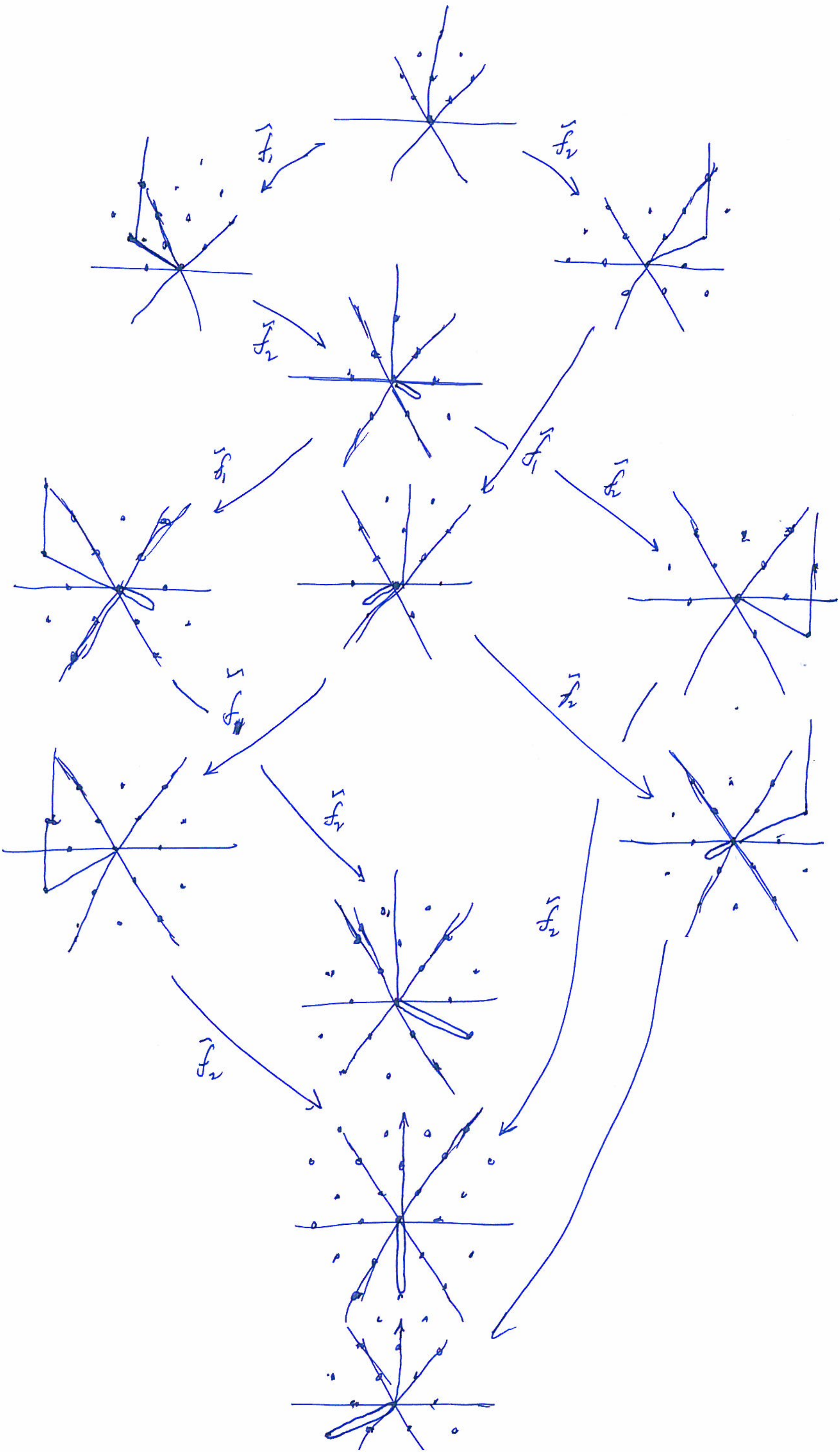
Type SL_n or GL_n

Let p_1, p_2, p_3 be



~~Thm~~, If $p_\lambda = p_1 \cdots p_1 p_2 \cdots p_2 \cdots$ then

$B(\lambda) = \{ p_T \mid T$ is the arabic reading of a column strict tableau of shape λ.



$\tilde{f}_1$
$\tilde{f}_2$
$\tilde{f}_2$
$\tilde{f}_1$
$\tilde{f}_1$
$\tilde{f}_2$
$\tilde{f}_1$
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$\tilde{f}_2$

and
$$\tilde{e}_i q = \begin{cases} p, & \text{if } q = \tilde{f}_i p \\ 0, & \text{otherwise} \end{cases}$$

If
$$\mathfrak{h}_{\mathbb{Z}}^{+} = \mathfrak{h}_{\mathbb{Z}} \cap \overline{C_0} \quad \text{and} \quad \mathfrak{h}_{\mathbb{Z}}^{++} = \mathfrak{h}_{\mathbb{Z}} \cap C_0$$
then
$$\begin{array}{ccc} \mathfrak{h}_{\mathbb{Z}}^{+} & \xrightarrow{\sim} & \mathfrak{h}_{\mathbb{Z}}^{++} \\ \lambda & \longmapsto & \rho+\lambda \end{array} \quad \text{as } \mathfrak{h}_{\mathbb{Z}}^{+}\text{-modules}.$$

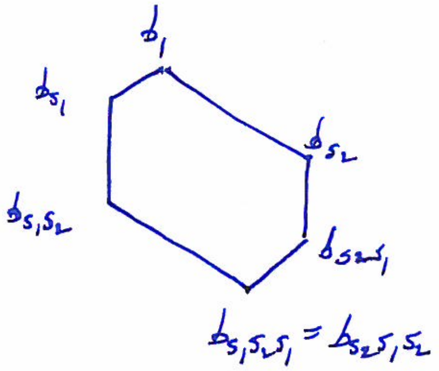
A <u>highest</u> <u>weight</u> <u>path</u> is $p \in B_{univ}$ with
$$\operatorname{im} p \subseteq C_0 - \rho.$$

p is a highest wt path $\Leftrightarrow$ $\tilde{e}_i p = 0$ for $i = 1, \ldots, n$.

Let $\lambda \in \mathfrak{h}_{\mathbb{Z}}^{+}$ and p_λ a hw path with $p_\lambda(1) = \lambda$
$$B(\lambda) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_d} p_\lambda \mid d \in \mathbb{Z}_{\geq 0}, 1 \leq i_1, \ldots, i_d \leq n \}.$$

<u>Theorem</u> (Littelmann) $B(\lambda)$ is a crystal and
$$s_\lambda = \sum_{p \in B(\lambda)} X^{p(1)}.$$

MV polytopes

$b =$ 

$per_{121}(b) = (l_1, l_2, l_3)$

$per_{212}(b) = (l_1', l_2', l_3')$

$b = Conv(\{b_w \mid w \in W_0\})$ with b_w the vertices of b.

$b_1 = 0$.

Let w_0 be the longest element of W_0 (Coxeter generators) and $\vec{i} = (i_1, \ldots, i_N)$ with $s_{i_1} \cdots s_{i_N} = w_0$ reduced.

The $\vec{i}$-perimeter of b is

$$per_{\vec{i}}(b) = (l_1, \ldots, l_N) \quad \text{with } l_k \beta = b_{s_{i_1} \cdots s_{i_k}} - b_{s_{i_1} \cdots s_{i_{k-1}}}$$

so that l is the distance from $b_{s_{i_1} \cdots s_{i_{k-1}}}$ to $b_{s_{i_1} \cdots s_{i_k}}$

β is the direction $\perp$ to $\mathfrak{h}^\beta$ separating $s_{i_1} \cdots s_{i_{k-1}} C_0 \,/\, s_{i_1} \cdots s_{i_k} C_0$ $\mathfrak{h}^\beta$.

Any other $per_{\vec{j}}(b)$ is obtained from $per_{\vec{i}}(b)$ by a sequence of transformations

$$R_{ij}^{ji}(l_k, l_{k+1}) = (l_{k+1}, l_k), \quad \text{if } s_i s_j = s_j s_i$$

$$R_{iji}^{jij}(l_k, l_{k+1}, l_{k+2}) = (l_k + l_{k+1} - \min(l_k, l_{k+2}), \min(l_k, l_{k+2}), l_{k+1} + l_{k+2} - \min(l_k, l_{k+2}))$$

if $s_i s_j s_i = s_j s_i s_j$.

The <u>crystal</u> <u>operator</u> $\hat{f}_{i_1} b$ is given by

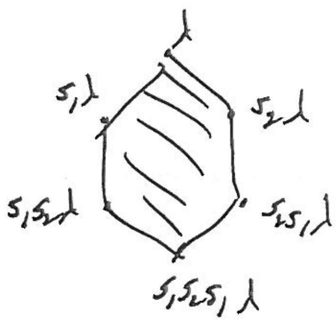
$$per_{\vec{i}}(\hat{f}_{i_1} b) = (l_1+1, l_2, l_3, \dots, l_N) \quad \text{if} \quad per_{\vec{i}}(b) = (l_1, l_2, \dots, l_N)$$

Let b_+ be given by $per_{\vec{i}}(b_+) = (0, 0, \dots, 0)$.

$$B(\infty) = \{\hat{f}_{i_1} \cdots \hat{f}_{i_d} b_+ \mid d \in \mathbb{Z}_{\geq 0},\ 1 \leq i_1, \dots, i_d \leq n\}$$

For $\lambda \in \mathfrak{h}^+_{\mathbb{Z}}$ let

$$B(\lambda) = \{b \in B(\infty) \mid \lambda + b \subseteq \mathrm{Conv}(W_0 \lambda)\}.$$



<u>Theorem</u> (Anderson-Kamnitzer)

$$s_\lambda = \sum_{b \in B(\lambda)} X^{\lambda + b_{w_0}}.$$

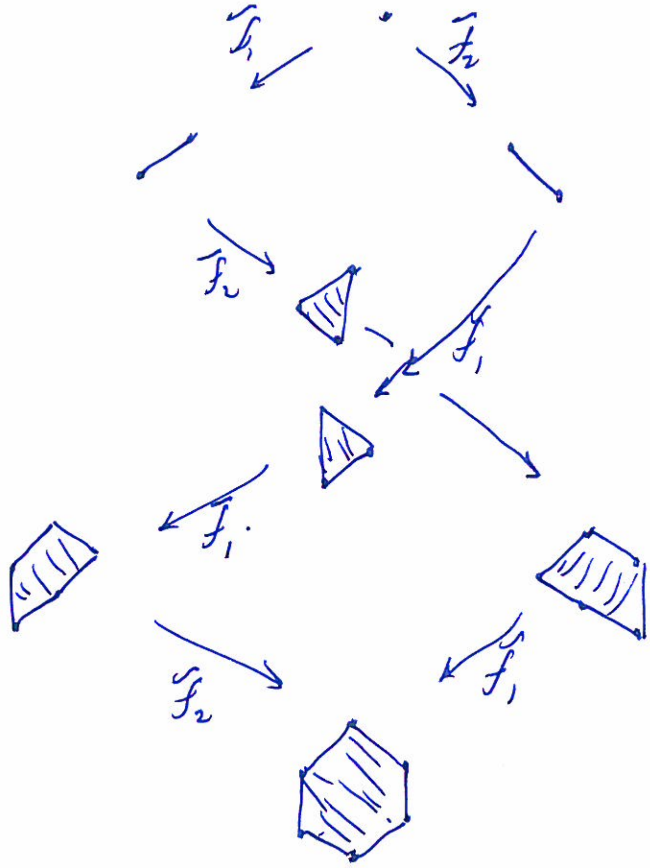
Theorem

Let $\lambda \in \mathfrak{h}^{\vee}_{\mathbb{Z}}$ with $\lambda \in \overline{C_0}$ and

$$B(\lambda) = \{ \text{MV polytopes } \delta \text{ with } \delta \subseteq \mathrm{Conv}(W_0 \lambda) \}.$$

The Weyl character is

$$s_\lambda = \sum_{\delta \in B(\lambda)} X^{\lambda - \delta_{w_0}}$$



<u>MV-cycles</u>

$$\mathbb{C}((t)) = \{a_{-\ell}t^{-\ell} + a_{-\ell+1}t^{-\ell+1} + \cdots \mid a_i \in \mathbb{C},\ \ell \in \mathbb{Z}\}$$
$$\cup\!|$$
$$\mathbb{C}[[t]] = \{a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i \in \mathbb{C}\} \xrightarrow{t=0} \mathbb{C}.$$

$G(\mathbb{C})$ is a complex reductive algebraic group

say $G = SL_3$.

$$G = SL_3(\mathbb{C}((t)))$$
$$\cup\!|$$
$$K = SL_3(\mathbb{C}[[t]]) \xrightarrow{t=0} SL_3(\mathbb{C})$$

G/K is ~~the~~ <u>loop</u> <u>Grassmannian</u>.

$$G = \bigsqcup_{\lambda \in \mathfrak{h}^+_{\mathbb{Z}}} K t_\lambda K \quad \text{and} \quad G = \bigsqcup_{\mu \in \mathfrak{h}_{\mathbb{Z}}} U^- t_\mu K$$

where $t_\lambda = h_\lambda(t)$ and $U^- = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \;\middle|\; * \in \mathbb{C}((t)) \right\}$.

The <u>Mirkovic-Vilonen intersections</u> are

$$K t_\lambda K \cap U^- t_\mu K.$$

The <u>MV cycles</u> are the <u>irreducible components</u>

Z_b in $\mathrm{Irr}(K t_\lambda K \cap U^- t_\mu K)$.

Then the <u>Weyl character</u> is

$$s_\lambda = \sum_{\mu \in \mathfrak{h}_{\mathbb{Z}}} \mathrm{Card}(\mathrm{Irr}(\overline{K t_\lambda K \cap U^- t_\mu K}))\, X^\mu.$$

Define $\tilde{f}_i b$ by

$y_i(ct^k) Z_b$ has $Z_{\tilde{f}_i b}$ as a dense open subset.