Lecture 5: Crystals from KLR and preprojective algebras

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Dynkin diagrams

G a complex reductive algebraic group

corresponds to

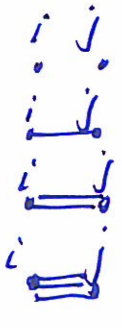
$(\mathfrak{h}_{\mathbb{Z}}, W_0)$ with $\quad \mathfrak{h}_{\mathbb{Z}}$ a free $\mathbb{Z}$-module
W_0 a finite subgroup of $GL(\mathfrak{h}_{\mathbb{Z}})$ generated by reflections

Let C_0 be a fundamental region for the action of W_0 on $\mathfrak{h}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}$. Let

$\mathfrak{h}^{\alpha_1}, \ldots, \mathfrak{h}^{\alpha_n}$ be the walls of C_0

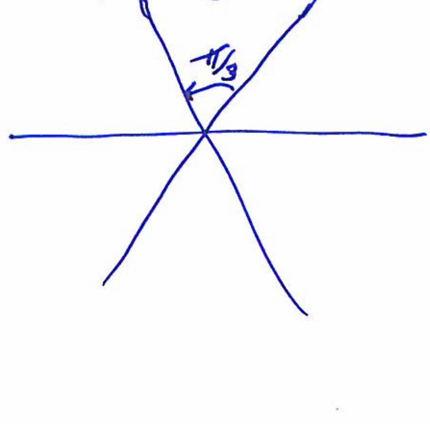
Make a graph with vertices $1, \ldots, n$

and



if $\mathfrak{h}^{\alpha_i} \measuredangle \mathfrak{h}^{\alpha_j}$ is $\quad \pi/2$, $\pi/3$, $\pi/4$, $\pi/6$.

Example Type SL_3



The quiver Hecke algebra: Choose an orientation.

$Q =$  $\quad I = \{\text{colors}\} = \{\text{vertices of } Q\}$.

$$\mathfrak{h}_{\mathbb{Z}}^{\geq 0} = \mathbb{Z}_{\geq 0}\text{-span}\{\alpha_i \mid i \in I\}.$$

Let $\gamma \in \mathfrak{h}_{\mathbb{Z}}^{\geq 0}$. The quiver Hecke algebra R_γ has generators $y_1, \ldots, y_d$, e_u for $u \in I^\gamma$, $\psi_1, \ldots, \psi_{d-1}$

and relations $y_i y_j = y_j y_i$, $e_u e_v = \delta_{uv} e_u$, $1 = \sum_{u \in I^\gamma} e_u$

... and more ....

where $I^\gamma = \left\{ u = (u_1, \ldots, u_d) \text{ sequences of colors with } \alpha_{u_1} + \cdots + \alpha_{u_d} = \gamma \right\}$

with $\mathbb{Z}$-grading

$$\deg(e_u) = 0, \quad \deg(y_i) = 2, \quad \deg(\psi_r e_u) = \begin{cases} -2, & \text{if } u_r = u_{r+1} \\ 1, & \text{if } u_r \text{ — } u_{r+1} \\ 0, & \text{if } u_r \quad u_{r+1} \end{cases}$$

Let $R = \bigoplus_{\gamma \in \mathbb{Z}_{\geq 0}} R_\gamma$

M a $\mathbb{Z}$graded R-module so that $M = \bigoplus_{j \in \mathbb{Z}} M[j]$

Define

$$\mathrm{char}(M) = \sum_{j \in \mathbb{Z}} \sum_{u \in I^\gamma} \dim(e_u M[j])\, q^j f_{u_1} \cdots f_{u_d}.$$

(generating function in noncommutative f_i, $i \in I$).

<u>Theorem</u> (Khovanov-Lauda/Rouquier)

$$\text{char}: \begin{matrix}\text{Grothendieck}\\ \text{group}\end{matrix}\left\{\begin{matrix}\text{fin. dim'l } \mathbb{Z}\text{-graded}\\ R \text{ modules}\end{matrix}\right\} \longrightarrow U_q\mathfrak{n}^- \quad \left(\begin{matrix}\text{quantum}\\ \text{group}\end{matrix}\right)$$

$$\begin{matrix}\text{simple}\\ R\text{-mods}\end{matrix} \quad L_b \longmapsto \text{char}(L_b) \quad \left(\begin{matrix}\text{canonical}\\ \text{basis}\end{matrix}\right)$$

Define $\tilde{f}_i b$ by

$$L_{\tilde{f}_i b} = \text{head}\left(\text{Ind}_{R_{\alpha_i}\otimes R_\gamma}^{R_{\alpha_i+\gamma}}(L_b)\right)$$

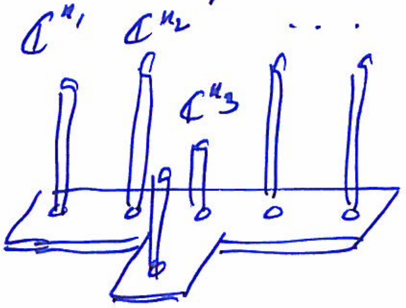
<u>Theorem</u> As directed graphs with labels $\xrightarrow{\tilde{f}_i}$

$$\left\{\begin{matrix}\text{simple } \mathbb{Z}\text{-graded}\\ R\text{-modules } L_b\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{MV polytopes}\\ b\end{matrix}\right\}.$$

Preprojective algebras

$$Q = \begin{tikzcd} \circ & \circ \arrow[l] & \circ \arrow[l] \arrow[d] \arrow[r] & \circ & \circ \arrow[l] \\ & & \circ & & \end{tikzcd} \qquad \overline{Q} = \begin{tikzcd} \circ \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] & \circ \arrow[l, shift left] \\ & & \circ \arrow[u, shift left] & & \end{tikzcd}$$

Idea: Replace beads by vector spaces.



$\mathbb{C}^{n_j}$ corresponds to n_j beads on runner j.

The data of

a vector space for each vertex

a linear transformation for each edge

is a <u>representation</u> of Q (or $\overline{Q}$).

In the case of $\overline{Q}$ require

$$\sum_{i \xrightarrow{a} \cdot \, \in Q} a^* a = \sum_{\cdot \xrightarrow{a} i \, \in Q} a a^*, \quad \text{for each } i \in I.$$

<u>Example</u>: <u>Type GL_n</u> $\quad Q = \circ \xleftarrow{a_1} \circ \xleftarrow{a_2} \circ \xleftarrow{a_3} \circ \leftarrow \circ \leftarrow \circ \cdots \xleftarrow{a_{n-2}} \circ$

$$\overline{Q} = \circ \underset{a_1^*}{\overset{a_1}{\rightleftarrows}} \circ \underset{a_2^*}{\overset{a_2}{\rightleftarrows}} \circ \rightleftarrows \cdots\cdots \underset{a_{n-2}^*}{\overset{a_{n-2}}{\rightleftarrows}} \circ$$

and we require

$$a_1 a_1^* = 0, \quad a_{i-1}^* a_{i-1} = a_i a_i^* \quad \text{for } i = 2, \ldots, n-2, \quad a_{n-2}^* a_{n-2} = 0$$

Let

$$\Lambda_b = \left\{ \text{isomorphism classes of representations of } \overline{Q} \text{ satisfying (PP)} \right\}$$

<u>Theorem</u> ~~There is an~~ As ~~gra~~ directed graphs with labels $\xrightarrow{f_i}$,

$$\left\{ \begin{array}{c} \text{irreducible components} \\ \Lambda_b \text{ of } \Lambda \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{MV polytopes} \\ b \end{array} \right\}$$