Soergel bimodules and Rouquier complexes, Algebra Geometry Topology Seminar, Melbourne Univ. 19 October 2012

Coxeter groups

Fix $(m_{ij}) \in M_n(\mathbb{Z}_{\geq 0} \cup \{\infty\})$ with $m_{ij} = m_{ji}$.

The Coxeter group associated to (m_{ij}) is the group W_0 given by generators $s_1, \ldots, s_n$ and relations $s_i^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$.

Let $x \in W_0$. A reduced word for x is

$x = s_{i_1} \cdots s_{i_\ell}$ with $s_{i_1}, \ldots, s_{i_\ell} \in \{s_1, \ldots, s_n\}$ and ℓ minimal

If $x = s_{i_1} \cdots s_{i_\ell}$ is a reduced word define

$\ell(x) = \ell$ and $y \leq x$ if $y = s_{i_1}^{a_1} \cdots s_{i_\ell}^{a_\ell}$ with $a_1, \ldots, a_\ell \in \{0,1\}$

The Hecke algebra of W_0 is the $\mathbb{Z}[v, v^{-1}]$-algebra $\mathcal{H}$ generated by $T_{s_1}, \ldots, T_{s_n}$ with relations

$$T_{s_i}^2 = (v^{-1} - v) T_{s_i} + 1 \quad \text{and} \quad \underbrace{T_{s_i} T_{s_j} T_{s_i} \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_{s_j} T_{s_i} T_{s_j} \cdots}_{m_{ij} \text{ factors}}.$$

Theorem $\mathcal{H}$ has $\mathbb{Z}[v, v^{-1}]$-basis $\{T_x \mid x \in W_0\}$

Let $\bar{\ }: \mathcal{H} \to \mathcal{H}$ be the $\mathbb{Z}$-algebra automorphism given by

$$\overline{T_{s_i}} = T_{s_i}^{-1} \quad \text{and} \quad \bar{v} = v^{-1}.$$

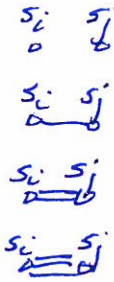
The Kazhdan-Lusztig basis of $\mathcal{H}$ is $\{C_x \mid x \in W_0\}$ characterized by

$$\overline{C_x} = C_x \quad \text{and} \quad C_x = T_x + \sum_{y < x} p_{yx} T_y \quad \text{with } p_{yx} \in v\mathbb{Z}[v]$$

Conjecture: $p_{yx} \in v \mathbb{Z}_{\geq 0}[v]$.

Reflection groups

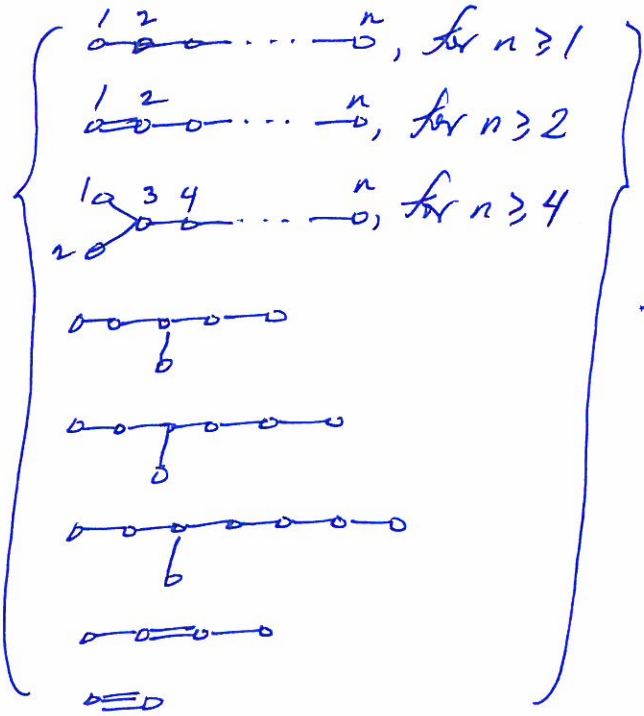
Let



indicate

$$s_i s_j = s_j s_i$$
$$s_i s_j s_i = s_j s_i s_j$$
$$s_i s_j s_i s_j = s_j s_i s_j s_i$$
$$s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i.$$

Theorem The map



$\longrightarrow$ $\left\{\begin{array}{l}\text{Irreducible finite}\\ \text{crystallographic}\\ \text{reflection groups } W_0\end{array}\right\}$

$$\Gamma \longmapsto W_0 = \langle s_1, \ldots, s_n \mid s_i^2 = 1, \text{ relations in } \Gamma \rangle$$

is a bijection.

A crystallographic reflection group is a $\mathbb{Z}$-reflection group.

A $\mathbb{Z}$-reflection group is a pair $(W_0, \mathfrak{h}_{\mathbb{Z}}^*)$ with

$\mathfrak{h}_{\mathbb{Z}}^*$ a free $\mathbb{Z}$-module

W_0 a subgroup of $GL(\mathfrak{h}_{\mathbb{Z}}^*)$ generated by reflections.

A Euclidean reflection group is an $\mathbb{R}$-reflection group.

$\mathbb{Z}$-graded R-bimodules

A *reflection* is an element $s \in GL_n(\mathbb{C})$ such that

s is conjugate to $\begin{pmatrix} \xi & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{pmatrix}$ with $\xi \neq 1$.

Let s be a reflection in $GL(\mathfrak{h}^*)$. Then

$$\mathfrak{h}^* = (\mathfrak{h}^*)^s \oplus \mathbb{C}\alpha_s, \quad \text{where}$$

$\alpha_s \in \mathfrak{h}^*$ with $s\alpha_s = \xi\alpha_s$ and $(\mathfrak{h}^*)^s = \{\mu \in \mathfrak{h}^* \mid s\mu = \mu\}$.

Let $x_1, \ldots, x_n$ be a basis of $\mathfrak{h}^*$. Define

$$R = H_T(pt) = \mathbb{C}[x_1, \ldots, x_n] = S(\mathfrak{h}^*) = \bigoplus_{k \in \mathbb{Z}} S^k(\mathfrak{h}^*)$$

with $S^k(\mathfrak{h}^*) = 0$ if $k < 0$ and $\deg(f) = 2k$ if $f \in S^k(\mathfrak{h}^*)$

A *$\mathbb{Z}$-graded module* is an R-module M with a decomposition

$$M = \bigoplus_{\ell \in \mathbb{Z}} M_\ell \quad \text{such that} \quad S^k(\mathfrak{h}^*) \cdot M_\ell \subseteq M_{2k+\ell}.$$

The *i-shift of M* is

$$M(i) = \bigoplus_{\ell \in \mathbb{Z}} M_\ell = \bigoplus_{\ell' \in \mathbb{Z}} M(i)_{\ell'} \quad \text{with} \quad M(i)_{\ell'} = M_{\ell' - i}.$$

Define

$$B_s = R \otimes_{R^s} R(1) \quad \text{where } R^s = \{f \in R \mid sf = f\}.$$

B_s is a $\mathbb{Z}$-graded R-bimodule with right R-basis

$$\{1 \otimes 1, \tfrac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)\}$$

Soergel bimodules

Let $s_{i_1}, \dots, s_{i_\ell}$ be reflections. The Bott-Samelson bimodule is

$$BS(s_{i_1}, \dots, s_{i_\ell}) = B_{s_{i_1}} \otimes_R B_{s_{i_2}} \otimes_R \cdots \otimes_R B_{s_{i_\ell}}.$$

$\cancel{BS(s_{i_1}, \dots}$ which has right R-basis

$$\{ c_{s_{i_1}}(a_1) \cdots c_{s_{i_\ell}}(a_\ell) \mid a_1, \dots, a_\ell \in \{0, \infty\}\}$$

where $c_{s_{i}}(0) = 1 \otimes 1$ and $c_s = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$.

The category of Soergel bimodules is the (additive Karoubian sub)category $\mathcal{B}$ of $\mathbb{Z}$-graded R-bimodules generated by Bott-Samelson bimodules and their shifts.

$\mathcal{B}$ has a monoidal structure given by

$$M \cdot N = M \otimes_R N.$$

The split Grothendieck group $K_{split}(\mathcal{B})$ is generated by symbols $[B]$ for $B \in \mathcal{B}$ and with relations

$$[B] = [B'] + [B''] \quad \text{if} \quad B \simeq B' \oplus B'' \text{ in } \mathcal{B}.$$

$K_{split}(\mathcal{B})$ is a $\mathbb{Z}[v, v^{-1}]$-module by

$$[M(i)] = v^i [M] \quad \text{and} \quad [M][N] = [M \cdot N]$$

makes $K_{split}(\mathcal{B})$ into a ring.

Soergel's categorification theorem

Theorem Let W_0 be a Coxeter group with generators $s_1, \ldots, s_n$ and $\mathfrak{h}^* = \operatorname{span}\{\alpha_{s_1}, \ldots, \alpha_{s_n}\}$ with W_0-action

$$s_i \lambda = \lambda - \langle \lambda, \alpha_{s_i}^\vee \rangle \alpha_{s_i}, \quad \text{where } \alpha_{s_i}^\vee(\alpha_{s_j}) = -2\cos\left(\frac{\pi}{m_{ij}}\right)$$

Then

$$\begin{aligned} K_{split}(\mathcal{B}) &\xrightarrow{\sim} \mathcal{H} \quad \text{(the Hecke algebra)} \\ B_x &\longmapsto C_x \end{aligned}$$

where $\{B_x \mid x \in W_0\}$ are the indecomposable objects of $\mathcal{B}$, inductively by

$$BS(s_{i_1}, \ldots, s_{i_\ell}) \simeq B_x \oplus \bigoplus_{y < x} (B_y)^{\oplus m_y}$$

if $x = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for x.

Rouquier complexes

$$F_s = (0 \longrightarrow B_s \longrightarrow R(1) \longrightarrow 0)$$
$$f \otimes g \longmapsto fg$$

The monoidal structure on $\mathcal{B}$ induces a monoidal structure on $K^b(\mathcal{B})$, the homotopy category of bounded complexes in $\mathcal{B}$.

Theorem (Rouquier?)

$F_{s_{i_1}} \cdots F_{s_{i_\ell}}$ does not depend on the reduced word.

A <u>Rouquier complex</u> is a minimal subcomplex

$$F_w \subseteq F_{s_{i_1}} \cdots F_{s_{i_\ell}}.$$

Game: To control $B_s B_x$, control $F_s B_x$ where B_x is regarded as a complex concentrated in degree 0.

<u>Remark: Moment graph view on B_s.</u>

Let $H_T(\mathbb{P}^1) = \left\{ \begin{array}{c} f_1 \\ | \\ f_s \end{array} \;\middle|\; f_1, f_s \in R \text{ and } f_1 - f_s \in \alpha_s R \right\}$

with $\begin{array}{c} f_1 \\ | \\ f_s \end{array} \cdot \begin{array}{c} g_1 \\ | \\ g_s \end{array} = \begin{array}{c} f_1 g_1 \\ | \\ f_s g_s \end{array}, \quad \begin{array}{c} f_1 \\ | \\ f_s \end{array} + \begin{array}{c} g_1 \\ | \\ g_s \end{array} = \begin{array}{c} f_1 + g_1 \\ | \\ f_s + g_s \end{array}$

$p \begin{array}{c} f_1 \\ | \\ f_s \end{array} = \begin{array}{c} p f_1 \\ | \\ s(p) f_s \end{array}$ and $\begin{array}{c} f_1 \\ | \\ f_s \end{array} p = \begin{array}{c} f_1 p \\ | \\ f_s p \end{array}$

Then

$$R \otimes_{R^s} R \xrightarrow{\sim} H_T(\mathbb{P}^1)$$

$$1 \otimes 1 \longmapsto \begin{array}{c} 1 \\ | \\ 1 \end{array}$$

$$\tfrac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \longmapsto \begin{array}{c} \alpha_s \\ | \\ 0 \end{array}$$