Affine Hecke algebras and Clifford Theory, Groups and Combinatorics

The affine braid groups

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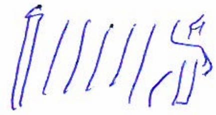
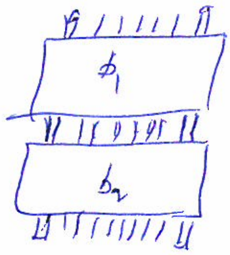
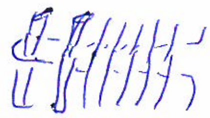
The affine braid group B_k is generated by

$T_0 =$  $T_i =$  $T_k =$ 

and product

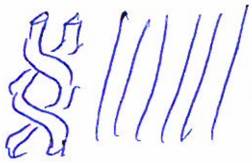
$b_1 b_2 =$ 

Move right pole around back to the left so that

$T_0 =$ , $T_i =$ , $T_k =$ 

let

$$Y_k = T_1^{-1} T_2^{-1} \cdots T_{k-1}^{-1} T_k T_{k-1} \cdots T_1 =$$ 

B_k^{ext} is B_k with an additional $P =$ 

and

$$P T_0 P^{-1} = T_0^{-1} T_k^{-1} T_0 T_k T_0, \quad P T_i P^{-1} = T_i, \quad P T_k P^{-1} = T_0^{-1} T_k T_0^{+1}.$$

so that

$$B_k^{ext} = B_k \rtimes \mathbb{Z} = B_k \rtimes \mathcal{P} \text{ where } \mathcal{P} = \{P^j \mid j \in \mathbb{Z}\}$$

Clifford Theory

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Let G be a group acting on
R, a ring, by automorphisms.

G acts on the set of simple R-modules $\{R^\lambda\}$

$${}^{g}R^\lambda = R^\lambda \quad \text{with} \quad r\circ m = g^{-1}(r)m.$$

$$R \rtimes G = R\text{-span}\{t_g \mid g \in G\} \text{ with } t_{g_1}t_{g_2} = t_{g_1g_2},\ t_g r = g^{-1}(r)t_g$$

$$R^G = \{r \in R \mid g(r) = r\}.$$

~~Let~~ Fix R^λ. The <u>inertia group</u> of R^λ is

$$H = \{h \in G \mid {}^{h}R^\lambda \cong R^\lambda\}.$$

Fix $\varphi_h : R^\lambda \xrightarrow{\sim} {}^{h^{-1}}R^\lambda$ and define

$$\alpha : H \times H \to \mathbb{C}^\times \quad \text{by} \quad \varphi_{h_1}\varphi_{h_2} = \alpha(h_1,h_2)\varphi_{h_1h_2}.$$

Define

$$(\mathbb{C}H)_{\alpha^{-1}} = \mathbb{C}\text{span}\{c_h \mid h \in H\} \text{ with } c_{h_1}c_{h_2} = \alpha(h_1,h_2)^{-1}c_{h_1h_2}$$

$$(\mathbb{C}H)_\alpha = \mathbb{C}\text{-span}\{b_h \mid h \in H\} \text{ with } b_{h_1}b_{h_2} = \alpha(h_1,h_2)b_{h_1h_2}$$

Let H^μ be a simple $(\mathbb{C}H)_{\alpha^{-1}}$-module. Define

$R \rtimes H$ acts on $R^\lambda \otimes H^\mu$ by $rt_h(m\otimes n) = r\varphi_h m \otimes c_h n$

$(\mathbb{C}H)_\alpha$ acts on $(H^\mu)^* = \mathrm{Hom}_{\mathbb{C}}(H^\mu,\mathbb{C})$ by $(b_h\psi)(m) = \psi(c_h^{-1}m)$

$(\mathbb{C}H_\alpha)$ acts on R^λ by $b_h\circ m = \varphi_h m$.

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Define $R^{\lambda,\mu}$ and $R_{\lambda,\mu}$ by

$$R^{\lambda,\mu} = \mathrm{Ind}_{R\rtimes H}^{R\rtimes G}\left(R^{\lambda}\otimes H^{\mu}\right)$$

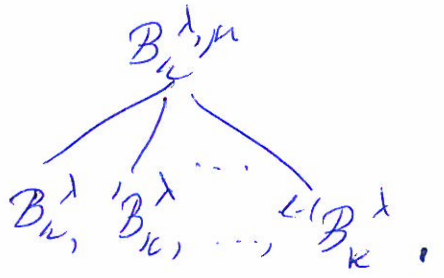
$$R^{\lambda} = \bigoplus_{\mu} R_{\lambda,\mu}\otimes (H^{\mu})^{*} \text{ as } \left(R^{G}, (\mathbb{C}H)_{\alpha}\right)\text{-modules.}$$

Theorem (a) $R^{\lambda\mu}$ are the simple $R\rtimes G$-modules

(b) $R_{\lambda\mu}$ are the simple R^{G}-modules

For B_k and B_k^{ext}: Let B_k^{λ} be a simple B_k-module.

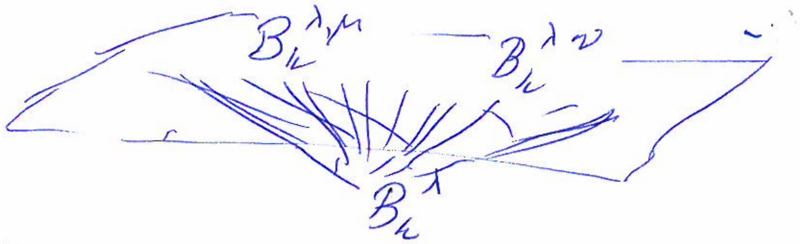
If $H = \ell\mathbb{Z}$



$\mu \in \mathbb{C}^{\times}$

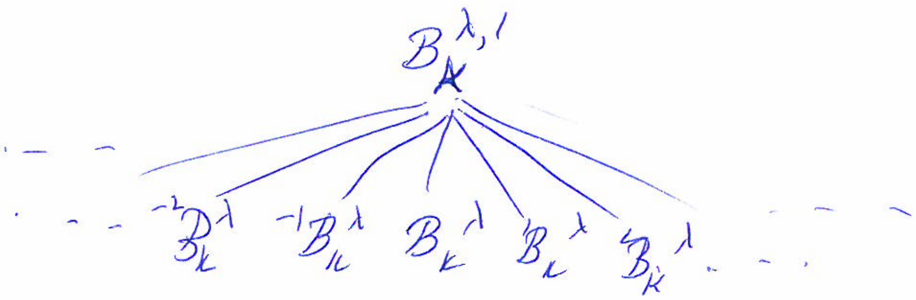
$\mathbb{Z}$-orbit of B_k^{λ}

If $H = \mathbb{Z}$



$\mu, \nu \in \mathbb{C}^{\times}$

If $H = \{1\} = 0\mathbb{Z} = \{0\}$



The Kazhdan-Lusztig indexing of B_k-modules

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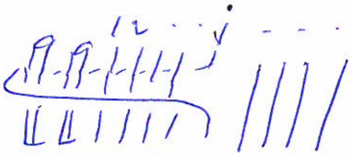
$Z_j =$ 

Then $Z_i Z_j = Z_j Z_i$ for $i,j \in \{1,\dots,k\}$

The Hecke algebra H_k of B_k is $\mathbb{C}B_k$ with

(*) $T_0^2 = (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})T_0 + 1$, $T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1$, $T_k^2 = (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})T_k + 1$

Kazhdan-Lusztig The simple H_k modules are (almost) characterized by the eigenvalues of Z_j:

There is $v_\lambda \in B_k^\lambda$ such that

$Z_1 v_\lambda = \lambda_1 v_\lambda, \dots, Z_k v_\lambda = \lambda_k v_\lambda$ with $\lambda_1, \dots, \lambda_k \in \mathbb{C}^\times$.

The affine Hecke algebra H_k^{ext} is $\mathbb{C}B_k^{ext}$ with relations (*).

Schur-Weyl duality for H_k^{ext}.

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$U = U_q \mathfrak{gl}_n$ the Drinfeld-Jimbo quantum group for GL_n.

U comes with R-matrices: U-module isomorphisms

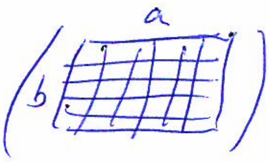
$$\check{R}_{MN} : M \otimes N \to N \otimes M \qquad \check{R}_{MN} =$$

The finite dim'l simple U-modules $L(\gamma)$.
are indexed by

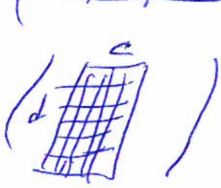
$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{Z}^n \quad \text{with } \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n.$$

Let

$$V = L(1, 0, \ldots 0) = L(\square)$$

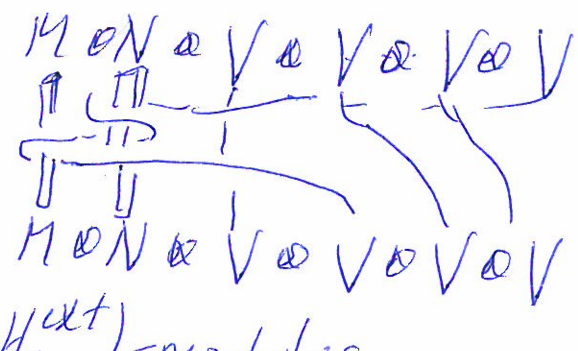
$$M = L(\underbrace{a, a, \ldots a}_{b}, 0, 0, \ldots 0) = L\Big($$


$$\Big)$$

$$N = L(\underbrace{c, \ldots c}_{d}, 0, 0, \ldots 0) = L\Big($$


$$\Big)$$

Then H_k^{ext} acts on $M \otimes N \otimes V^{\otimes k}$



As (U, H_k^{ext})-modules

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\gamma} L(\gamma) \otimes B_k^{\gamma}.$$

To compare indexings, find eigenvalues of Z_i
and use Clifford theory.