Modular forms, Hecke algebras and G_2-flags

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Flags in the 7-dimensional representation

For the group G_2:

$$G_2(R) = \mathrm{Aut}_{R\text{-alg}}(\mathbb{O} \otimes_{\mathbb{Z}} R)$$

where

$$\mathbb{O} = \mathbb{Z}\text{-span}\left\{ \begin{array}{l} l_1, l_2, \ldots, l_7,\ \tfrac{1}{2}l_1+\tfrac{1}{2}l_2+\tfrac{1}{2}l_3+\tfrac{1}{2}l_5, \\ \tfrac{1}{2}l_0+\tfrac{1}{2}l_1+\tfrac{1}{2}l_2+\tfrac{1}{2}l_4,\ \tfrac{1}{2}l_0+\tfrac{1}{2}l_1+\tfrac{1}{2}l_3+\tfrac{1}{2}l_7,\ \tfrac{1}{2}l_0+\tfrac{1}{2}l_1+\tfrac{1}{2}l_5+\tfrac{1}{2}l_6 \end{array} \right\}$$

Recall that if $u \in \mathbb{O}$ ~~then~~ with $u = \xi_0 l_0 + \cdots + \xi_7 l_7$

$$\bar{u} = \xi_0 l_0 - \xi_1 l_1 - \cdots - \xi_7 l_7, \quad Tr(u) = u + \bar{u} = 2\xi_0$$

$$N(u) = u\bar{u} = \xi_0^2 + \xi_1^2 + \cdots + \xi_7^2, \quad \langle x, y\rangle = Tr(x\bar{y})$$

Let k be a field, $\mathrm{char}(k) \neq 2$. Let

$$M = \mathbb{O} \otimes k \quad \text{and} \quad M_0 = l_0^{\perp} = \{x \in M \mid \langle x, e_0\rangle = 0\}.$$

Fact 1: If $x \in M_0$ then $N(x) = -x^2$.

Fact 2: $M_0 = \{x \in M \mid x \notin k l_0 \text{ and } x^2 \in k l_0\}$.
$= \{x \in M \mid x \notin k \text{ and } x^2 \in k\}$.

Fact 3: If $g \in \mathrm{Aut}(\mathbb{O} \otimes k) = G_2(k)$ then $gM_0 \subseteq M_0$

Fact 4: M_0 is the 7 dimensional irreducible $G_2(k)$-module.

Fact 5: If $g \in G_2(k)$ and $x \in M$ then

$$\overline{g(x)} = g(\bar{x}), \quad N(gx) = N(x), \quad Tr(gx) = Tr(x)$$

since $\bar{x}$ is characterized by $x + \bar{x} \in k$ and $x - \bar{x} \in M_0$.

A <u>null line</u> is $V_1 = \mathrm{span}\{v\}$ with $v \in V$ and $N(v) = 0$.

A <u>null plane</u> is $V_2 = \mathrm{span}\{v_1, v_2\}$ with $xy = 0$ for $x, y \in V_2$.

Let

$$V_1 = kv_1, \text{ where } v_1 \in M_0 \text{ with } N(v_1) = -v_1^2 = 0$$

$$V_2 = kv_1 + kv_2, \text{ where } v_2 \in M_0 \text{ with } v_1 \cdot v_2 = 0.$$

$$V_3 = \{ y \in M_0 \mid xy = 0 \text{ for } x \in V_2 \}$$

$$V_4 = V_3^\perp$$

$$V_5 = V_2^\perp$$

$$V_6 = V_1^\perp$$

$$V_7 = M_0$$

so that $0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 \subseteq V_5 \subseteq V_6 \subseteq M_0$

is determined by the data of $V_1 \subseteq V_2$.

Define

$$P_1 = \mathrm{Stab}(V_1), \qquad P_2 = \mathrm{Stab}(V_2)$$

$$B = \mathrm{Stab}(V_1 \subseteq V_2) = \mathrm{Stab}(V_1 \subseteq V_2 \subseteq \cdots \subseteq V_6 \subseteq M_0).$$

B is a Borel subgroup of $G_2(k)$
and P_1 and P_2 are (two examples) of standard parabolic subgroups.

G_2 has class number 1 so that

$$G_2(\hat{\mathbb{Q}}) = G(\mathbb{Q})\,G(\hat{\mathbb{Z}})$$

in general, if G is defined over $\mathbb{Z}$

$$G(\hat{\mathbb{Q}}) = G(\mathbb{Q})g_1 G(\hat{\mathbb{Z}}) \sqcup \cdots \sqcup G(\mathbb{Q})g_h G(\hat{\mathbb{Z}})$$

and h is the <u>class number</u> of G.

Let G be a connected reductive algebraic group over $\mathbb{Q}$ which is anisotropic over $\mathbb{R}$,

W a $G(\mathbb{Q})$-module

K an open compact subgroup of $G(\hat{\mathbb{Q}})$.

The space of <u>algebraic modular forms of weight W and level K</u> on G is

$$M(W,K) = \{ F : G(\hat{\mathbb{Q}})/K \to W \mid F(\gamma g) = \gamma F(g) \text{ for } \gamma \in G(\mathbb{Q}) \}$$

~~Since~~ There

$$K = \prod_p J_p \quad \text{where} \quad J_p = \begin{cases} G(\mathbb{Z}_p), & \text{for } p \notin S \\ \text{parahoric}, & \text{for } p \in S. \end{cases}$$

so that

$$\begin{array}{ccc} G(\mathbb{Z}_p) & \xrightarrow{p=0} & G(\mathbb{F}_p) \\ \cup\!\mid & & \cup\!\mid \\ J_p = \Phi^{-1}(P) & \longrightarrow & P \end{array} \qquad \text{and}$$

$$G(\mathbb{Z}_p)/J_p \longleftrightarrow G(\mathbb{F}_p)/P.$$

Goal: Study the Hecke algebra action on $M(W,K)$.

WHAT Hecke algebra? WHAT action?

Recall:

Let G be a finite group, H_1 and H_2 subgroups of G.

Let W_1 be a representation of H_1, $W_1 : H_1 \to GL(V_1) = GL_{n_1}(\mathbb{C})$

Let W_2 be a representation of H_2, $W_2 : H_2 \to GL(V_2) = GL_{n_2}(\mathbb{C})$

Define

$$M = \left\{ f : G \to \mathrm{Hom}(V_1, V_2) \;\middle|\; f(h_1 g h_2) = W_1(h_1) f(g) W_2(h_2), \text{ for } \begin{array}{l} h_1 \in H_1 \text{ and} \\ h_2 \in H_2 \end{array} \right\}$$

Then

$$\mathrm{Hom}_G(\mathrm{Ind}_{H_1}^G(W_1), \mathrm{Ind}_{H_2}^G(W_2)) \cong M \text{ as vector spaces.}$$

Special case:

$\mathrm{Hom}_G(\mathrm{Ind}_{H_2}^G(W_2), \mathrm{Ind}_{H_2}^G(W_2))$ is an algebra, and

$\mathrm{Hom}_G(\mathrm{Ind}_{H_2}^G(\mathrm{triv}), \mathrm{Ind}_{H_2}^G(\mathrm{triv}))$ is the Hecke algebra of the pair (G, H_2).

Then M is a left $\mathrm{End}_G(\mathrm{Ind}_{H_1}^G(W_1))$-module and a right $\mathrm{End}_G(\mathrm{Ind}_{H_2}(W_2))$-module.

So $M(W,K)$ is a left $\mathrm{End}_{G(\hat{\mathbb{Q}})}(\mathrm{Ind}_{G(\mathbb{Q})}^{G(\hat{\mathbb{Q}})}(W))$-module and a right $\mathrm{End}_{G(\hat{\mathbb{Q}})}(\mathrm{Ind}_K^{G(\hat{\mathbb{Q}})}(\mathrm{triv}))$-module.

Here $\mathrm{End}_{G(\hat{\mathbb{Q}})}(\mathrm{Ind}_K^{G(\hat{\mathbb{Q}})}(\mathrm{triv}))$ is the <u>Hecke algebra</u>.