ICERM 18.02.2013

p-adic integrals to sums over crystals

$$G = \bigsqcup_{w \in W} IwI \qquad G = \bigsqcup_{v \in W} U^- v I$$

Decompose

$$G = \bigsqcup_{v,w \in W} IwI \cap U^- vI = \bigsqcup_{v,w \in W} \bigsqcup_{p \in \mathcal{P}(w)_v} C_p$$

where

$\mathcal{P}(w)_v$ = "crystal" = $\{$paths of type w that end in $v\}$

$$C_p \cong \mathbb{F}^{(\#\text{ of steps } \xrightarrow{+} \text{ in } p)} \times (\mathbb{F}^\times)^{(\#\text{ steps } \stackrel{+}{\vdash\!\!\rightarrow} \text{ in } p)}$$

Then

$$\int_G f(g)\,d\mu = \sum_{v,w \in W} \sum_{p \in \mathcal{P}(w)_v} \int_{C_p} f(g)\,d\mu$$

and, if $\mathbb{F} = \mathbb{F}_q$ then

$$\mathrm{Card}(C_p) = q^{(\#\text{steps } \xrightarrow{+} \text{ in } p)} \cdot (q-1)^{(\#\text{steps } \stackrel{+}{\vdash\!\!\rightarrow} \text{ in } p)}$$

G and subgroups

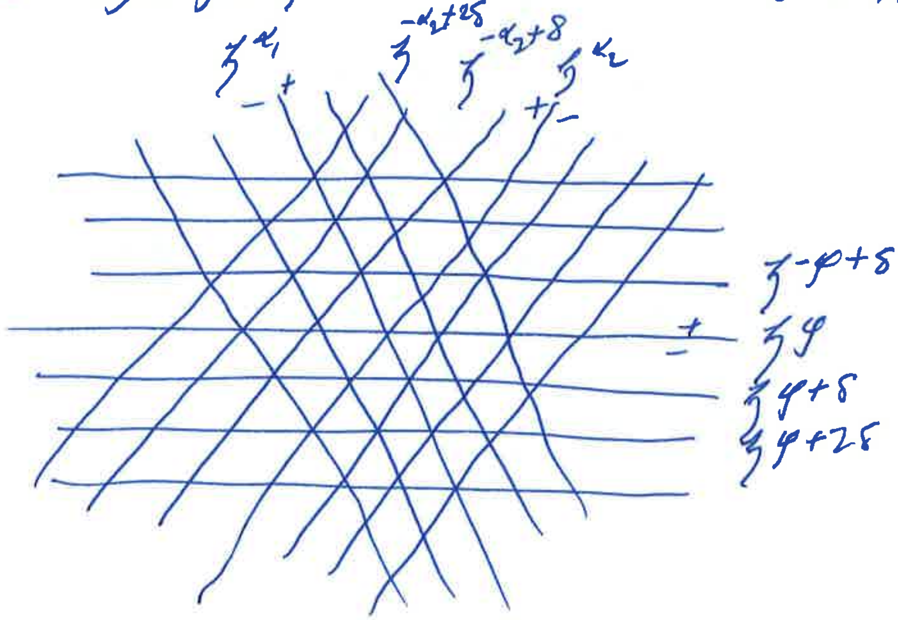
$$G = GL_3(\mathbb{F}_q((t))) \qquad K = GL_3(\mathbb{F}_q[[t]])$$

$$U^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{pmatrix} \right\} \subseteq GL_3(\mathbb{F}_q((t)))$$

$$B = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \right\} \subseteq GL_3(\mathbb{F}_q)$$

$$I = \left\{ \begin{pmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) \end{pmatrix} \in GL_3(\mathbb{F}_q[[t]]) \;\middle|\; \begin{pmatrix} p_{11}(0) & p_{12}(0) & p_{13}(0) \\ p_{21}(0) & p_{22}(0) & p_{23}(0) \\ p_{31}(0) & p_{32}(0) & p_{33}(0) \end{pmatrix} \in B \right\}$$

The <u>affine</u> <u>Weyl group</u> W is the set of alcoves in



The <u>Chevalley-Steinberg-Tits generators</u>, <u>elementary matrices</u>,

$$x_{\alpha_1+k\delta}(c) = \begin{pmatrix} 1 & ct^k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \cdots \quad \text{and}$$

$\mathcal{X}_{\alpha_1+k\delta} = \{ x_{\alpha_1+k\delta}(c) \mid c \in \mathbb{F}_q \}$ are the <u>root subgroups</u>

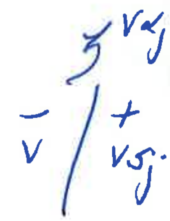
<u>The picture</u>

(1) The hyperplanes $\mathfrak{h}^{\beta+k\delta}$ are labeled by $\beta+k\delta$ with $\mathcal{X}_{\beta+k\delta} \subseteq I$.

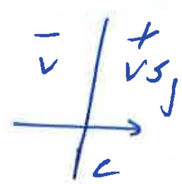
(2) The <u>periodic orientation</u> has the north star on the positive side of all hyperplanes.

(3) The root subgroup

$\mathcal{X}_{v\alpha_j} \subseteq U^-$ if and only if

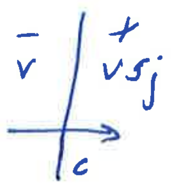


A <u>labeled green step of type j</u> is



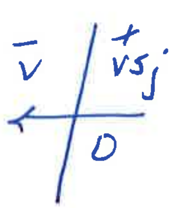
with $c \in \mathbb{F}_q$

A <u>labeled purple step of type j</u> is

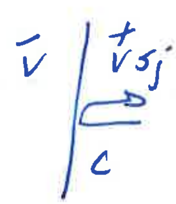


$c \in \mathbb{F}_q$

or



or



$c \in \mathbb{F}_q^\times$

<u>Theorem</u> Let $w \in W$. Let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w (minimal length path to w).

(a) The following map is a bijection

$$\left\{\begin{matrix}\text{labeled green paths}\\ \text{of type } i_1,\dots,i_\ell\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}I\text{-cosets in}\\ IwI\end{matrix}\right\}$$

$$\underbrace{(c_1,\dots,c_\ell)}_{\text{labels}} \longmapsto x_{i_1}(c_1)n_{i_1}^{-1}x_{i_2}(c_2)n_{i_2}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}I$$

(b) Let $v \in W$. The following map is a bijection.

$$\left\{\begin{matrix}\text{labeled purple paths}\\ \text{of type } i_1,\dots,i_\ell \text{ and}\\ \text{end } v\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}I\text{-cosets in}\\ IwI \cap U^- vI\end{matrix}\right\}$$

$$(\underbrace{p_1,\dots,p_\ell}_{\text{steps}};\ \underbrace{d_1,\dots,d_\ell}_{\text{labels}}) \longmapsto \underbrace{x_{\gamma_1}(d_1)\cdots x_{\gamma_\ell}(d_\ell)}_{\in U^-}n_v I$$

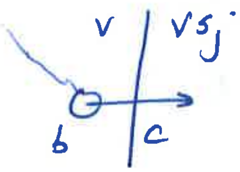
where $\mathfrak{h}^{\gamma_1},\dots,\mathfrak{h}^{\gamma_\ell}$ are the hyperplanes corresponding to the steps of p, $p_j =$  or $p_j =$ 

and

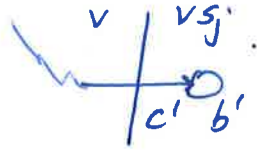
$$n_v = n_{j_1}^{-1}\cdots n_{j_k}^{-1} \quad \text{if } v = s_{j_1}\cdots s_{j_k} \text{ is reduced.}$$

Algorithm: labeled green to labeled purple paths

Case 0:

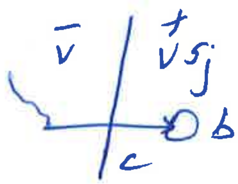


becomes

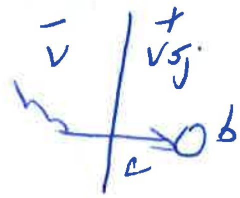


$$u^- n_v b\, x_j(c) n_j^{-1} = u^- n_v x_j(c') n_j^{-1} b' \qquad \begin{pmatrix} b, b' \in I \\ c, c' \in \mathbb{F}_q \end{pmatrix}$$

Case 1



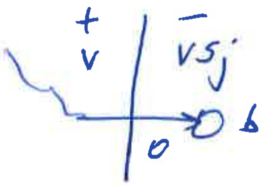
becomes



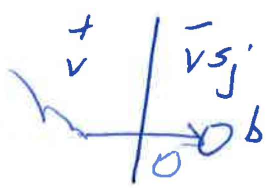
$$(c \in \mathbb{F}_q)$$

$$u^- n_v x_j(c) n_j^{-1} b = u^- x_{v\alpha_j}(c) n_{vs_j} b$$

Case 2

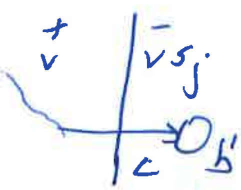


becomes

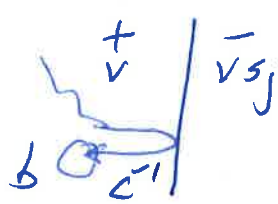


$$u^- n_v x_j(0) n_j^{-1} b = u^- x_{v\alpha_j}(0) n_{vs_j} b$$

Case 3



becomes

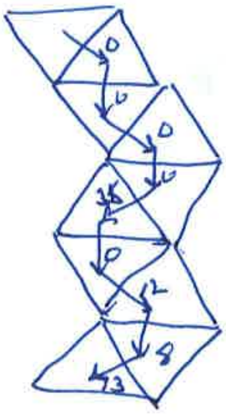


$$(c \in \mathbb{F}_q^\times)$$

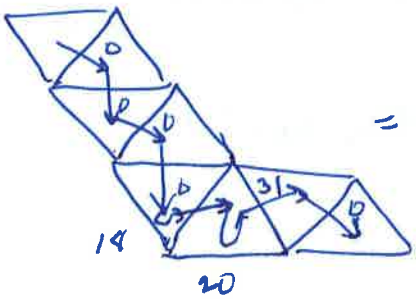
$$u^- n_v x_j(c) n_j^{-1} b' = u^- \underbrace{n_v x_{\alpha_j}(c^{-1})}\,\underbrace{x_{\alpha_j}(-c) h_{\alpha_j^\vee}(c) b'}$$

$$= u^- x_{-v\alpha_j}(c^{-1}) n_v b$$

Example $\mathbb{F}_q = \mathbb{F}_{37}$



$$= x_2(0)n_2^{-1}x_1(0)n_1^{-1}x_0(0)n_0^{-1}x_2(0)n_2^{-1}x_0(35)n_0^{-1}$$
$$\cdot x_1(0)n_1^{-1}x_0(12)n_0^{-1}x_2(8)n_2^{-1}x_0(13)n_0^{-1}$$

$=$ 

$$= x_{-\alpha_2}(0)x_{-\varphi}(0)x_{\alpha_2-\delta}(0)x_{-\varphi-\delta}(0)$$
$$x_{\alpha_1}(18)x_{-\alpha_2-2\delta}(0)x_{-\varphi-2\delta}(20)x_{-\alpha_1+\delta}(31)$$
$$x_{-\alpha_2-3\delta}(0)\,n_{s_2s_1s_0s_2s_1s_2s_0}\,b_{10}$$

where

$$b_{10} = \begin{pmatrix} 18+6t & 26 & 33 \\ -t^2 & 13+8t & 2+24t \\ 17t^2 & 25t & 34+t \end{pmatrix}$$

G = p-adic group

I = Iwahori subgroup

K = maximal compact

U^- = neg. unipotent subgroup

W = affine Weyl group

$\mathfrak{h}_{\mathbb{Z}}$ = coweight lattice

$\mathfrak{h}_{\mathbb{Z}}^+$ = dominant coweights

$$G = \bigsqcup_{t_\lambda \in \mathfrak{h}_{\mathbb{Z}}^+} K t_\lambda K \qquad \text{and} \qquad G = \bigsqcup_{t_\mu \in \mathfrak{h}_{\mathbb{Z}}} U^- t_\mu K$$

$$G = \bigsqcup_{t_\lambda \in \mathfrak{h}_{\mathbb{Z}}^+} \bigsqcup_{t_\mu \in \mathfrak{h}_{\mathbb{Z}}} (K t_\lambda K \cap U^- t_\mu K)$$

$$= \bigsqcup_{t_\lambda \in \mathfrak{h}_{\mathbb{Z}}^+} \bigsqcup_{t_\mu \in \mathfrak{h}_{\mathbb{Z}}} \bigsqcup_{p \in \mathcal{P}(\lambda)_\mu} C_p$$

with $C_p \simeq \mathbb{F}^{\left(\substack{\#\text{steps} \dashv\!\!\mid \\ \text{in } p}\right)} \times (\mathbb{F}^\times)^{\left(\substack{\#\text{steps} \mid\!\!\vDash \\ \text{in } p}\right)}$

$$\int_G f(g)\, d\mu = \sum_{t_\lambda \in \mathfrak{h}_{\mathbb{Z}}^+} \sum_{t_\mu \in \mathfrak{h}_{\mathbb{Z}}} \sum_{p \in \mathcal{P}(\lambda)_\mu} \int_{C_p} f(g)\, d\mu$$

$$\mathbb{F}_q((t)) = \left\{ a_{-\ell} t^{-\ell} + a_{-\ell+1} t^{-\ell+1} + \cdots \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z} \\ a_i \in \mathbb{F}_q \end{array} \right\}$$

$$\mathbb{F}_q[[t]] = \left\{ a_0 + a_1 t + a_2 t^2 + \cdots \;\middle|\; a_i \in \mathbb{F}_q \right\}$$

$$x_{\alpha_1 + k\delta}(c) = \begin{pmatrix} 1 & ct^k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad x_{\alpha_2 + k\delta}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & ct^k \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_{\varphi + k\delta}(c) = \begin{pmatrix} 1 & 0 & ct^k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_{-\alpha_1+k\delta}(c) = \begin{pmatrix} 1 & 0 & 0 \\ ct^k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad x_{-\alpha_2+k\delta}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & ct^k & 1 \end{pmatrix}$$

$$x_{-\varphi+k\delta}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ct^k & 0 & 1 \end{pmatrix}$$

The <u>root subgroups</u> are

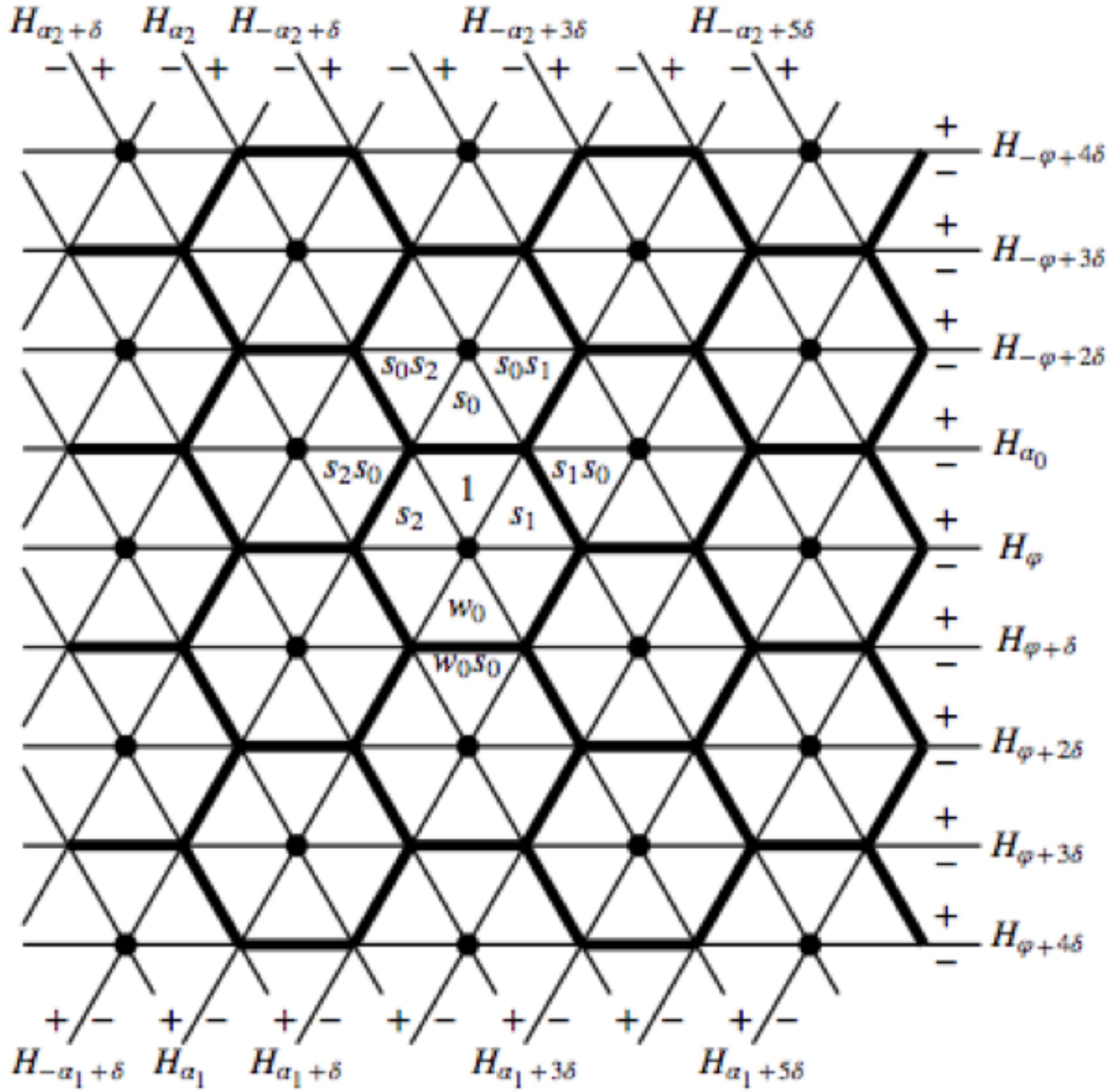
$$\mathcal{X}_{\beta+k\delta} = \left\{ x_{\beta+k\delta}(c) \,\middle|\, c \in \mathbb{F}_q \right\}$$

$$n_1^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad n_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$n_0^{-1} = \begin{pmatrix} 1 & 0 & t^{-1} \\ 0 & 1 & 0 \\ -t & 0 & 1 \end{pmatrix}$$

$$n_v = n_{j_1}^{-1} n_{j_2}^{-1} \cdots n_{j_r}^{-1}$$

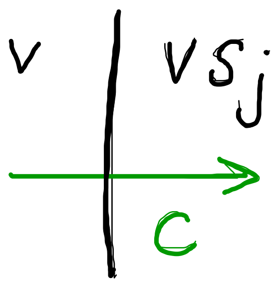
if $v = s_{j_1} s_{j_2} \cdots s_{j_r}$ is a reduced word for v


$H_{\alpha_2+\delta}$
H_{α_2}
$H_{-\alpha_2+\delta}$
$H_{-\alpha_2+3\delta}$
$H_{-\alpha_2+5\delta}$
$H_{-\varphi+4\delta}$
$H_{-\varphi+3\delta}$
$H_{-\varphi+2\delta}$
H_{α_0}
H_{φ}
$H_{\varphi+\delta}$
$H_{\varphi+2\delta}$
$H_{\varphi+3\delta}$
$H_{\varphi+4\delta}$
$H_{-\alpha_1+\delta}$
H_{α_1}
$H_{\alpha_1+\delta}$
$H_{\alpha_1+3\delta}$
$H_{\alpha_1+5\delta}$
s_0s_2
s_0s_1
s_0
s_2s_0
1
s_1s_0
s_2
s_1
w_0
w_0s_0

$$h_{\alpha_1^\vee}(c) = \begin{pmatrix} c & 0 & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad h_{\alpha_2^\vee}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}$$

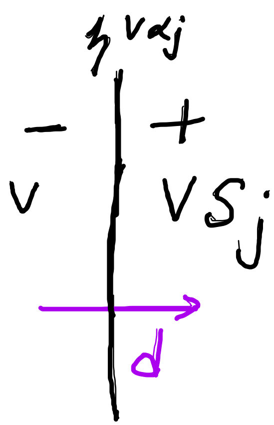
$$h_{\varphi^\vee}(c) = \begin{pmatrix} c^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}$$

A labeled nonfolded green step of type j is

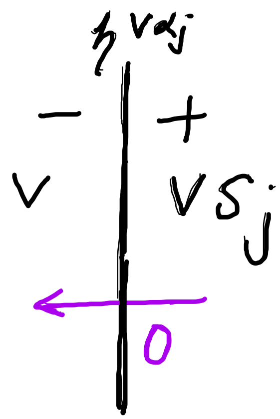


with $c \in \mathbb{F}_q$

A labeled folded purple step of type j



or



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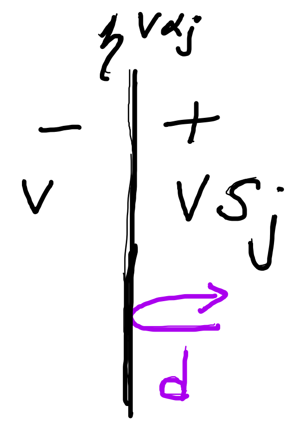
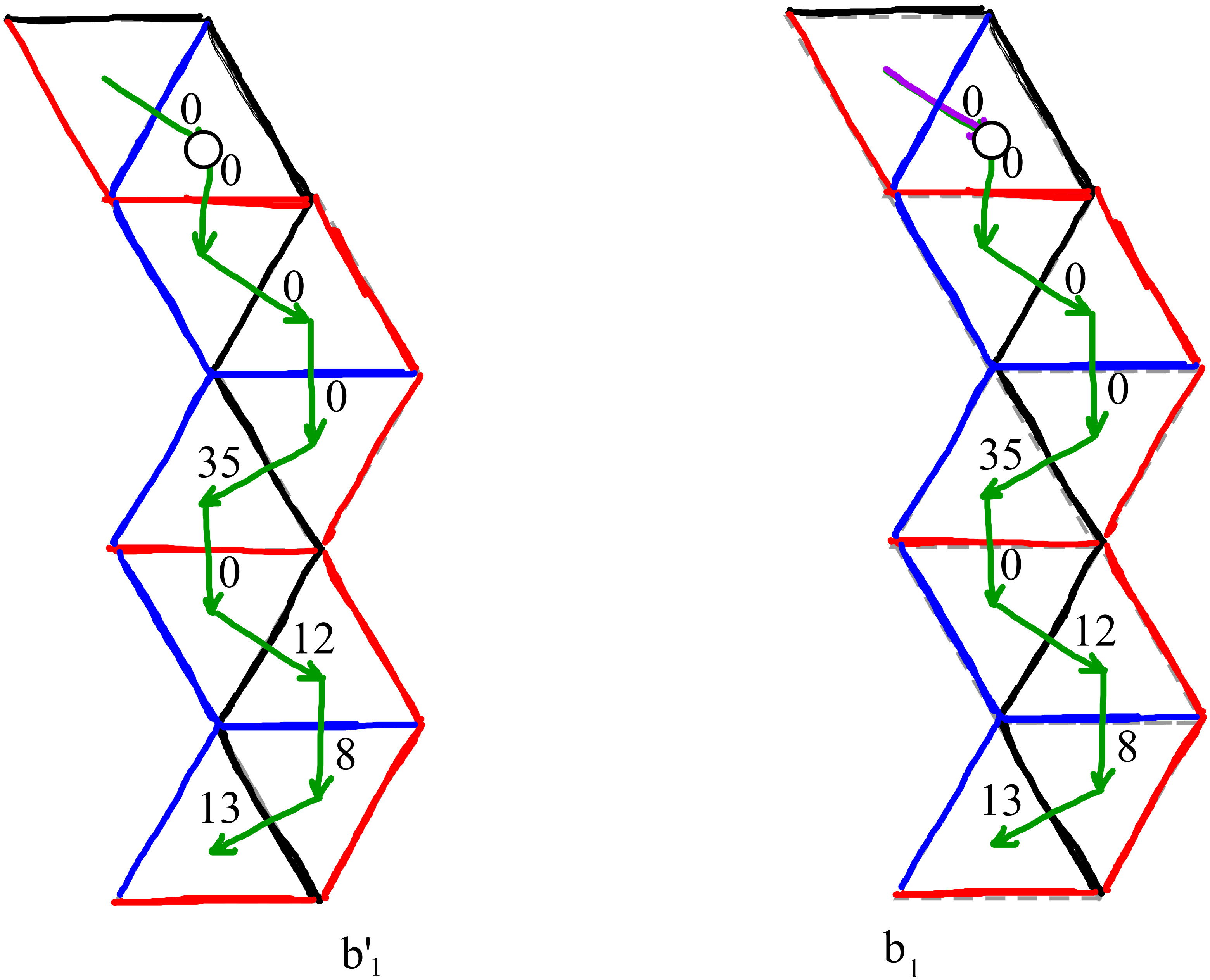


with $d \in \mathbb{F}_q$ with $d \in \mathbb{F}_q^\times$



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b'_1
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35
0
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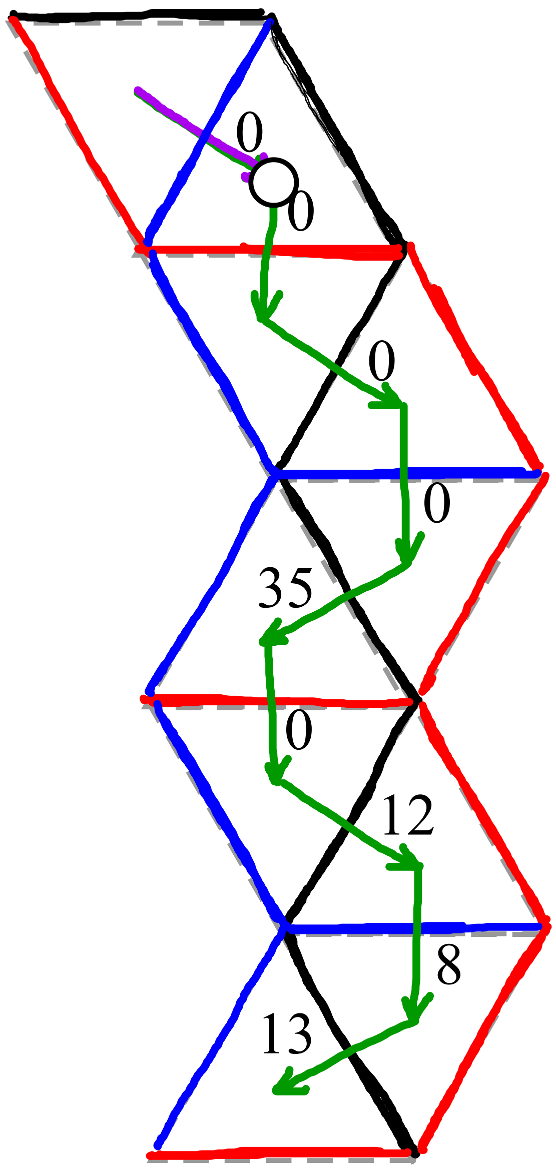
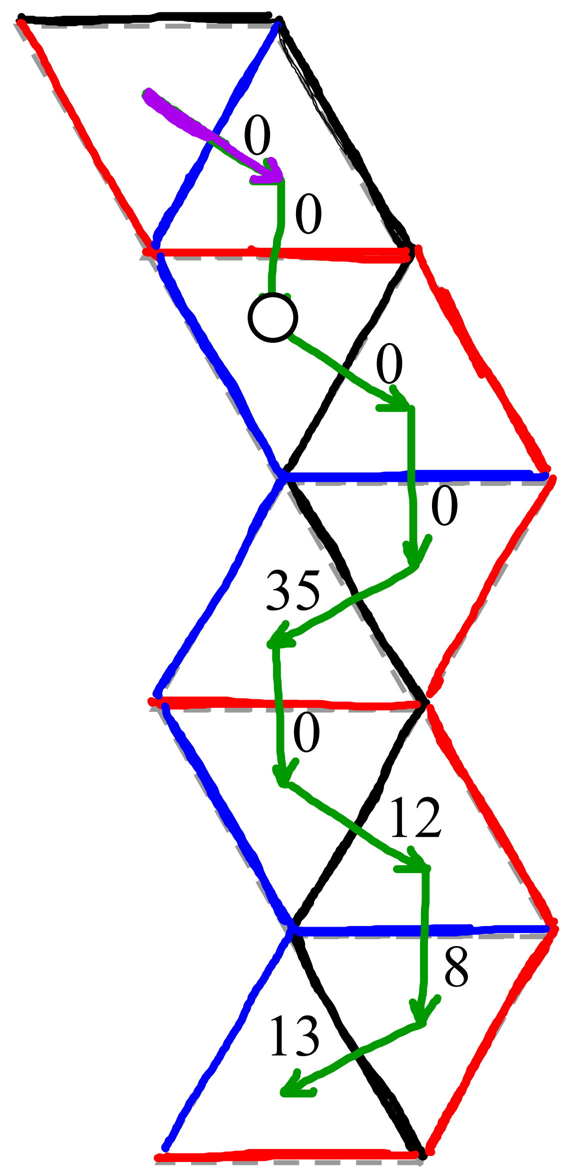
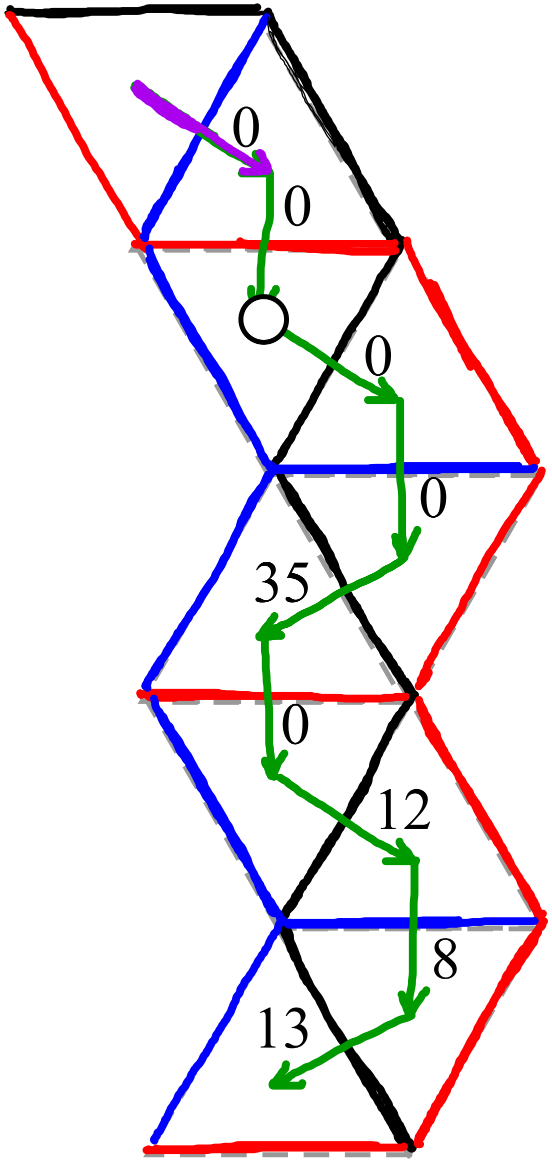
8
13
b_1



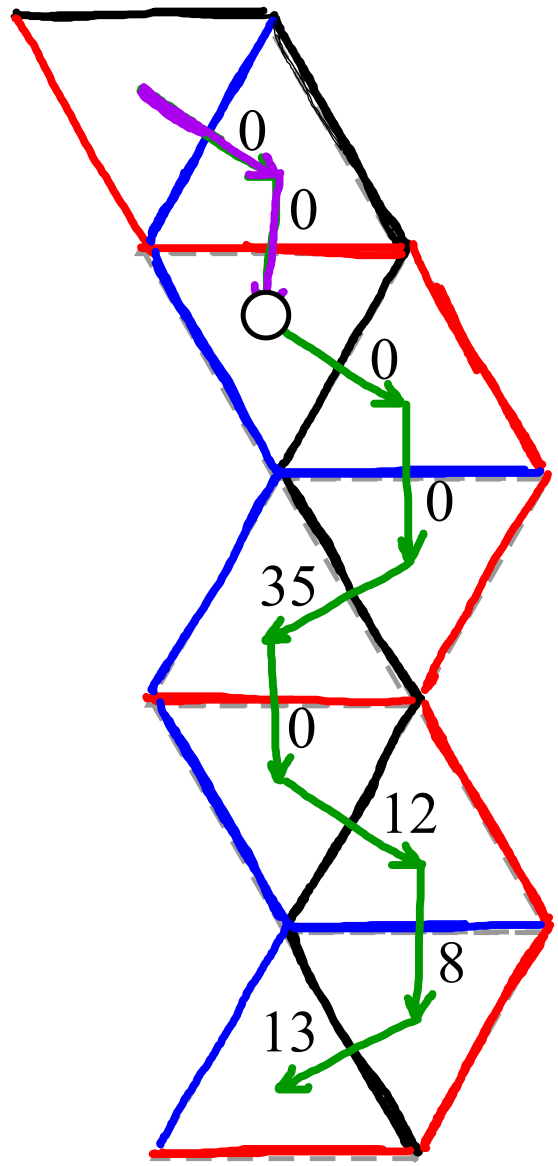
b_2



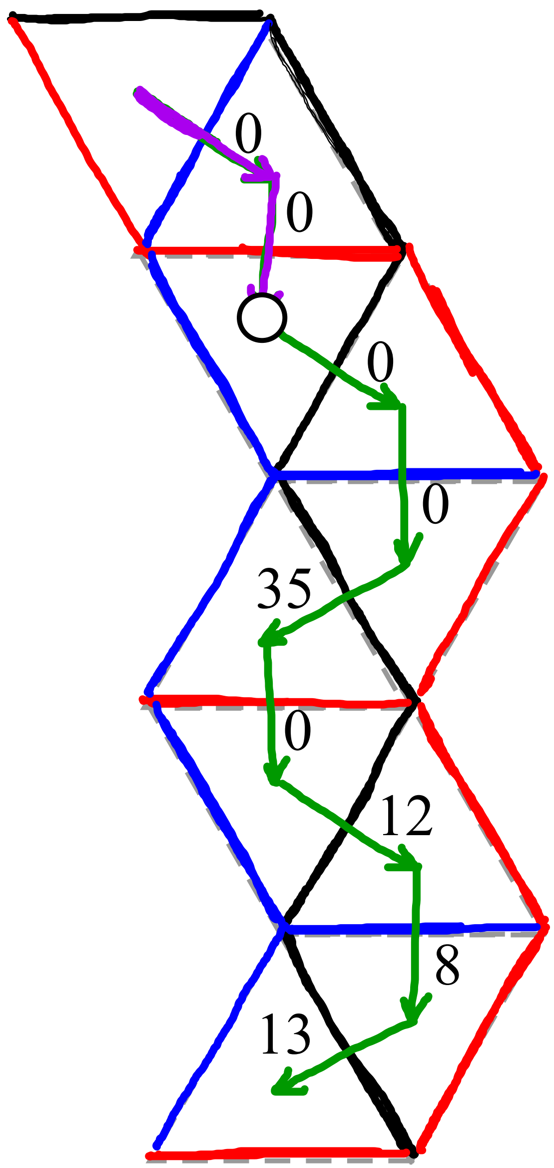
b'_2



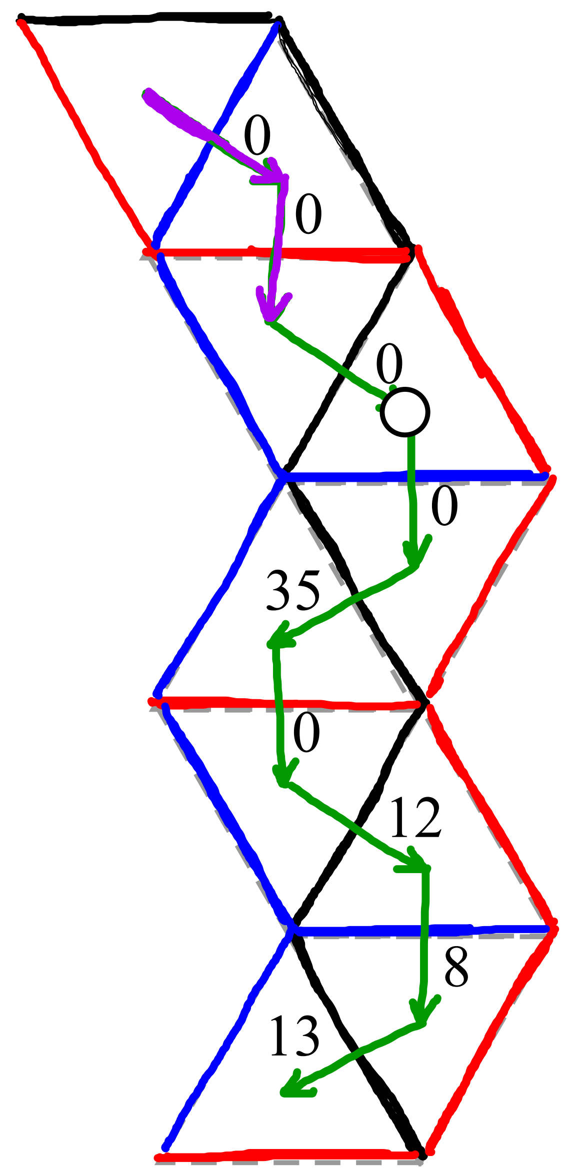
b'$_2$



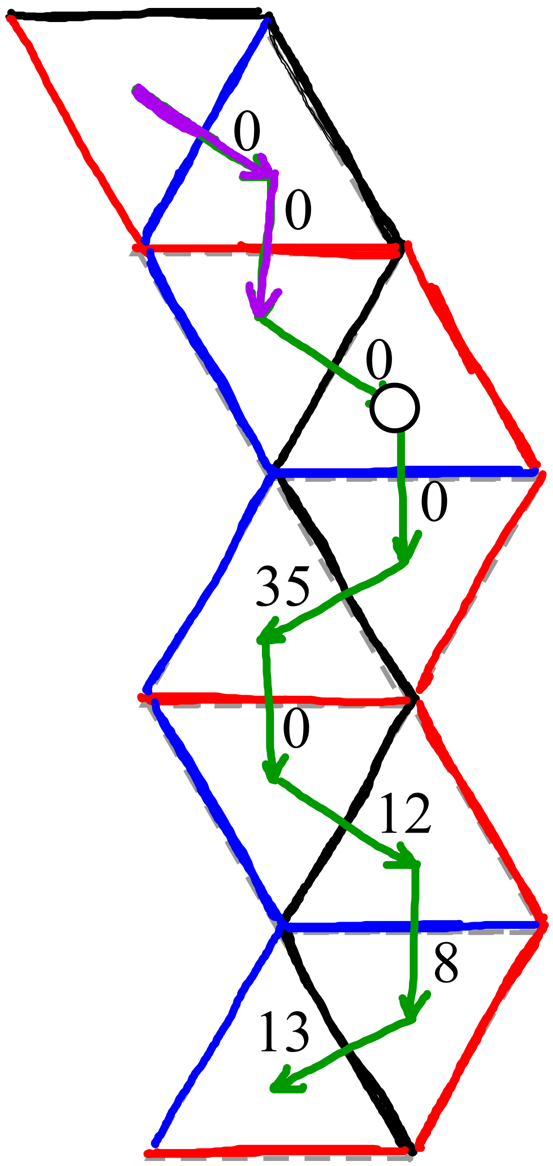
b$_3$



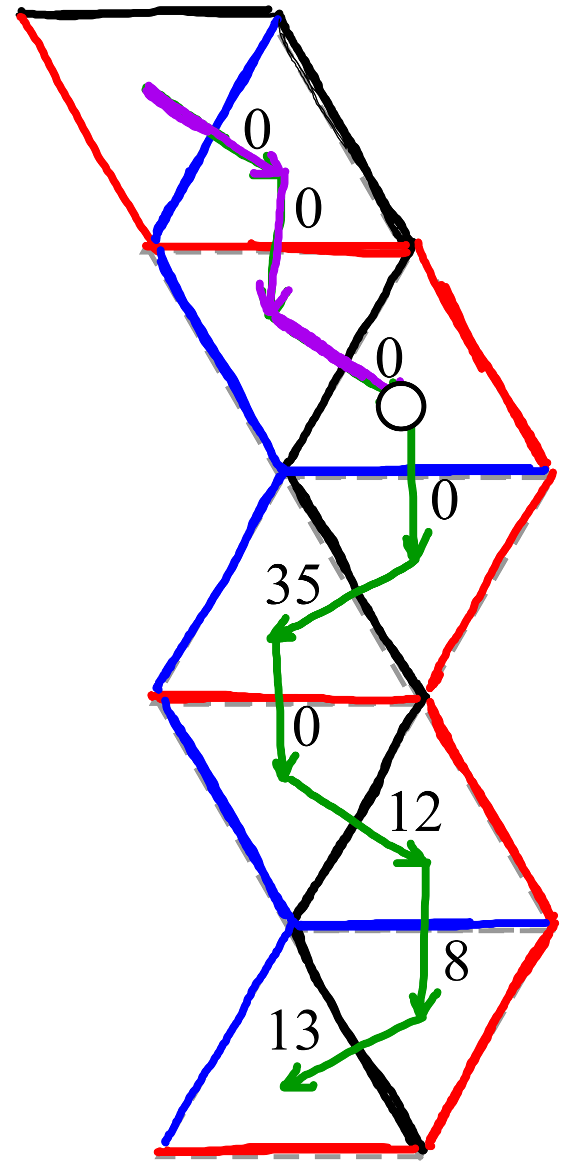
b_3



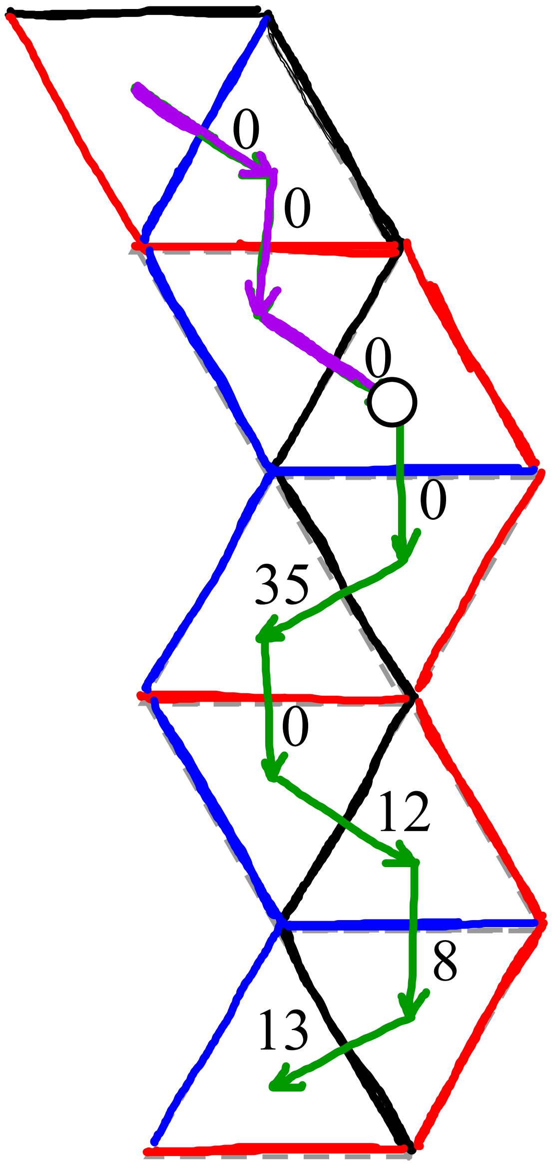
b'_3



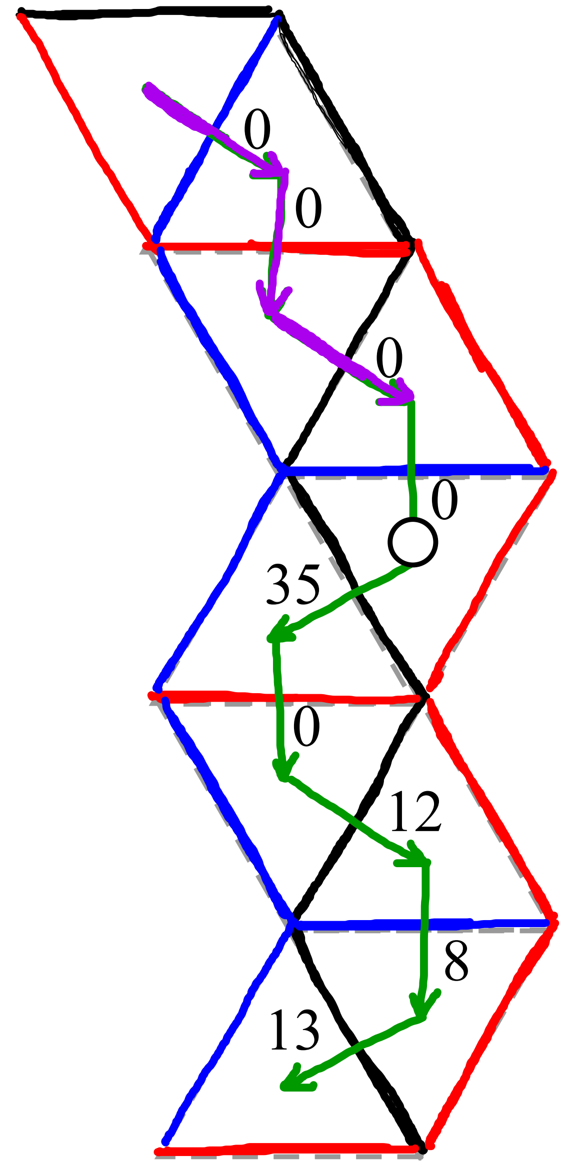
b'$_3$



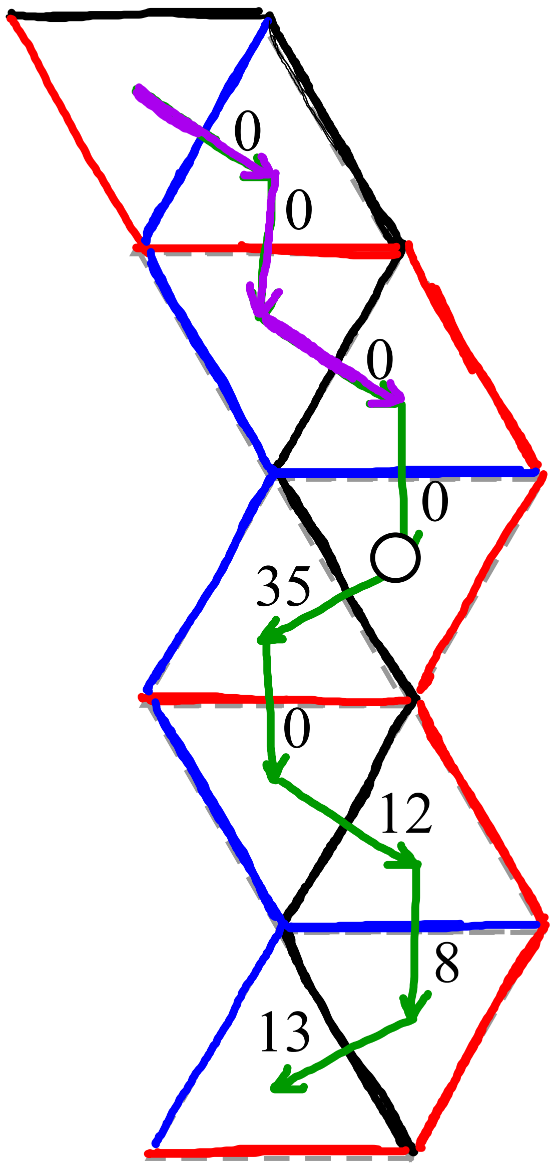
b_4



b_4



b'_4



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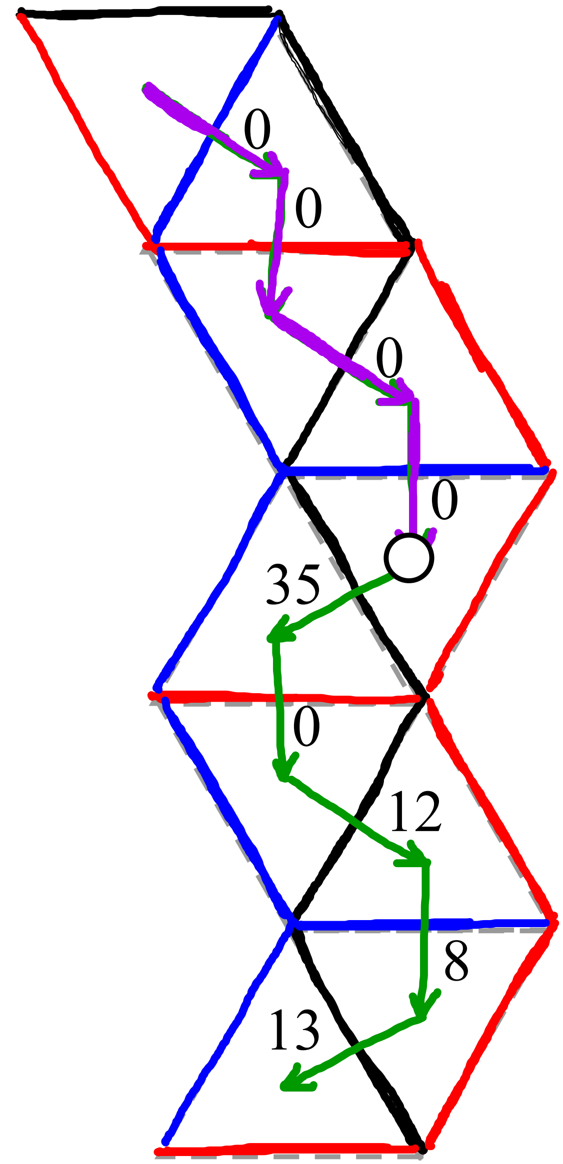
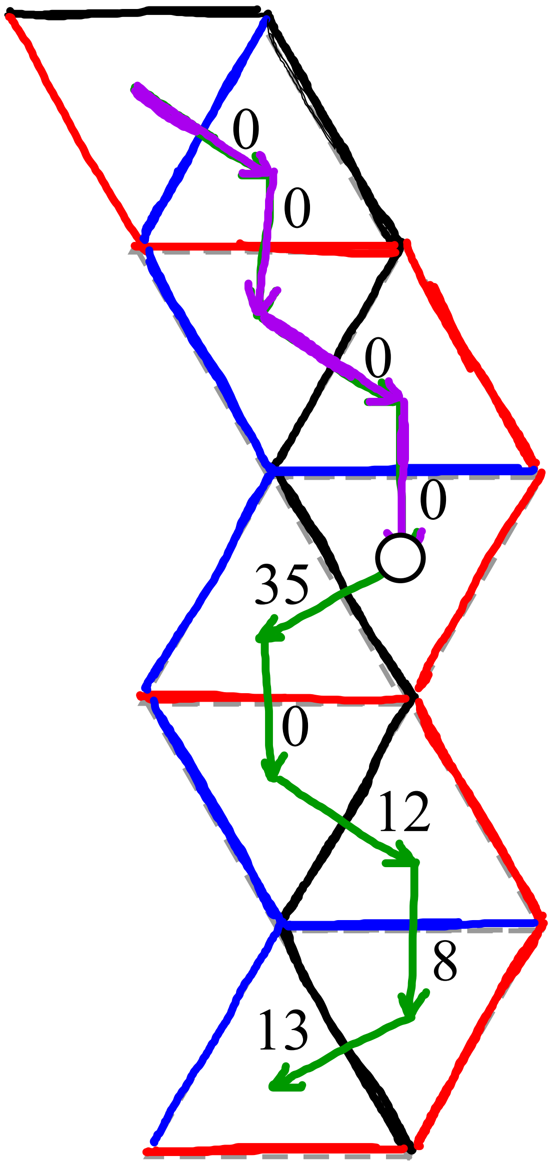
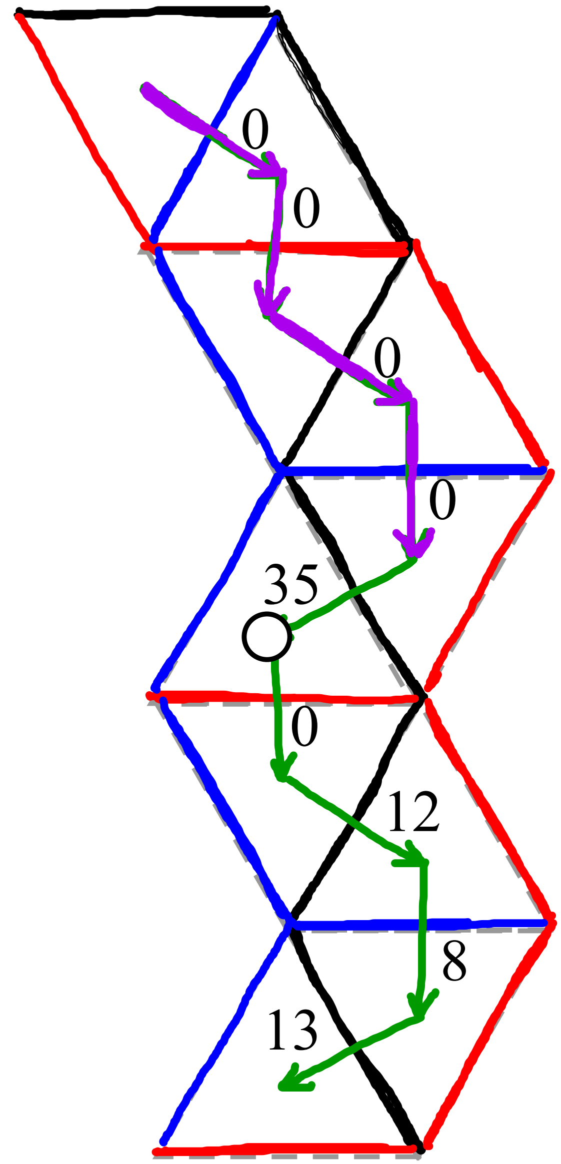


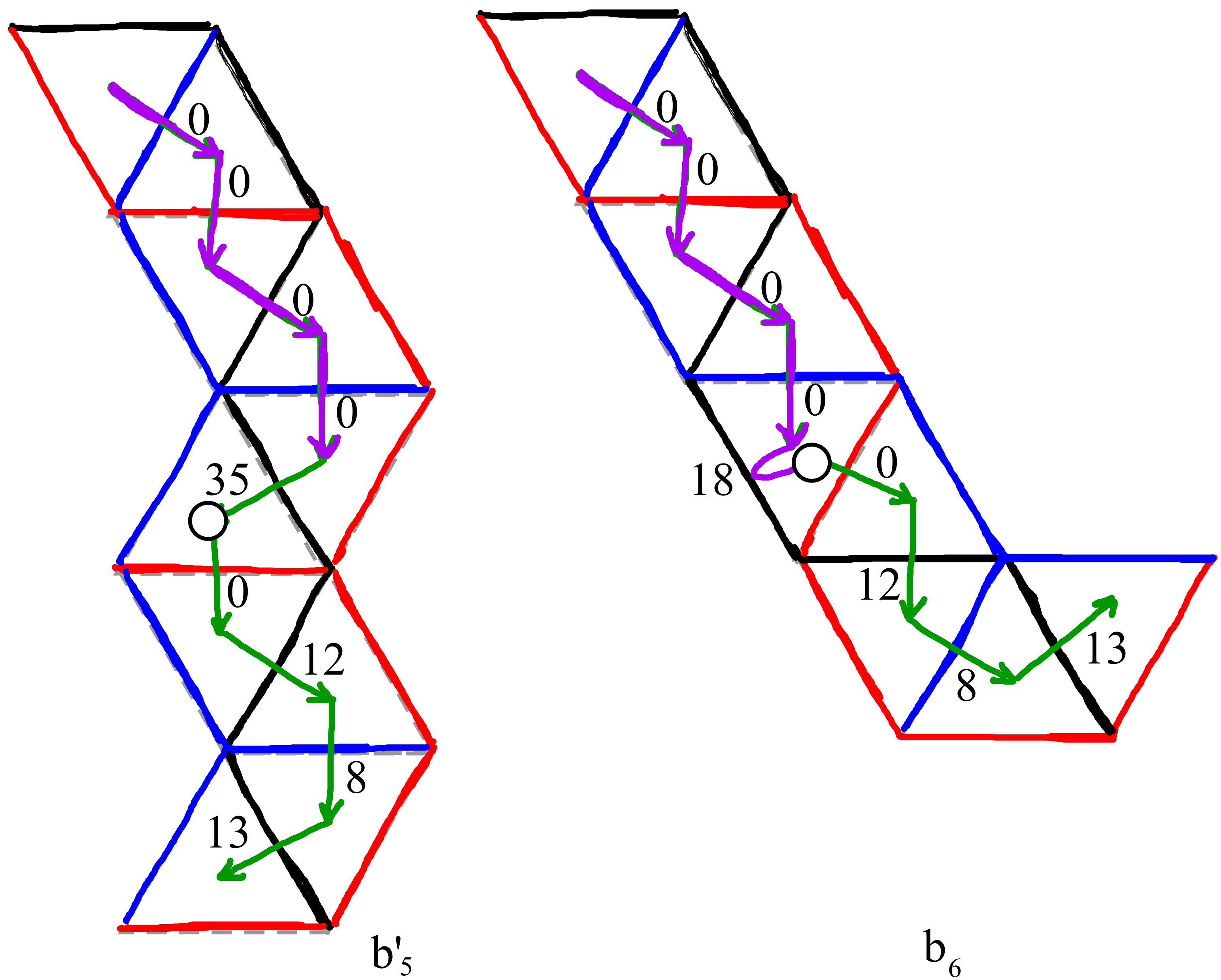
b_5



b_5



b'_5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b'5
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18
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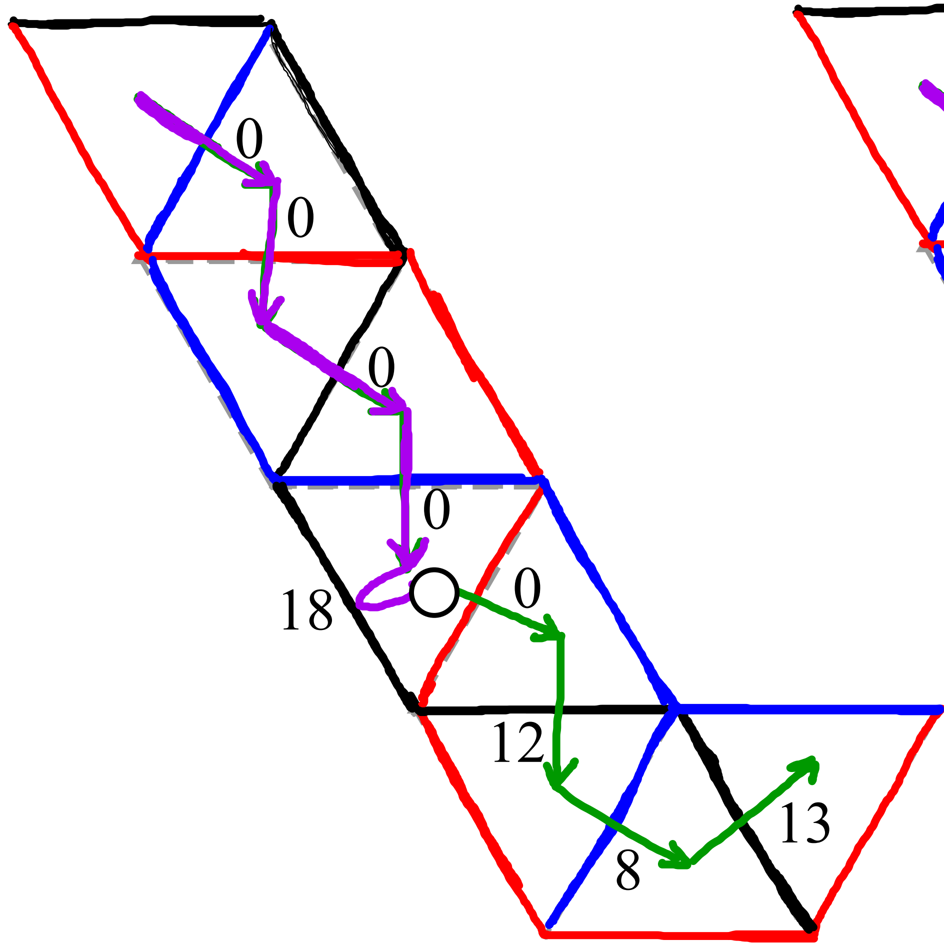
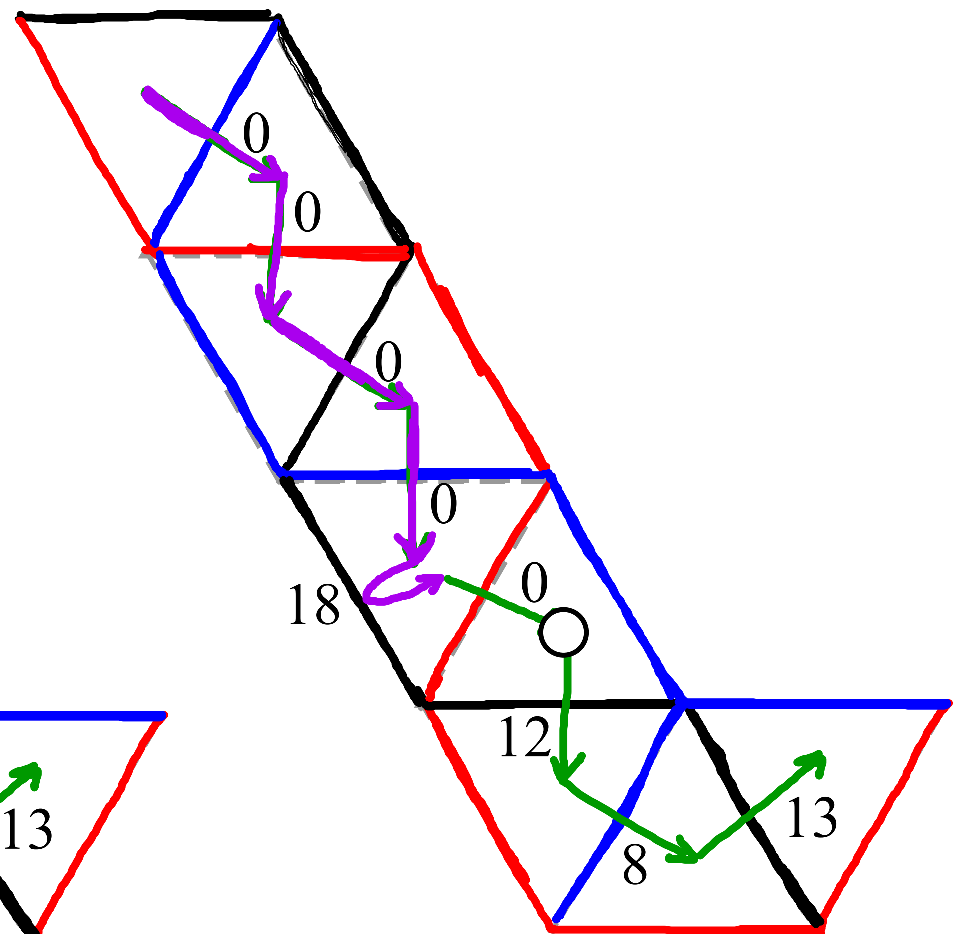
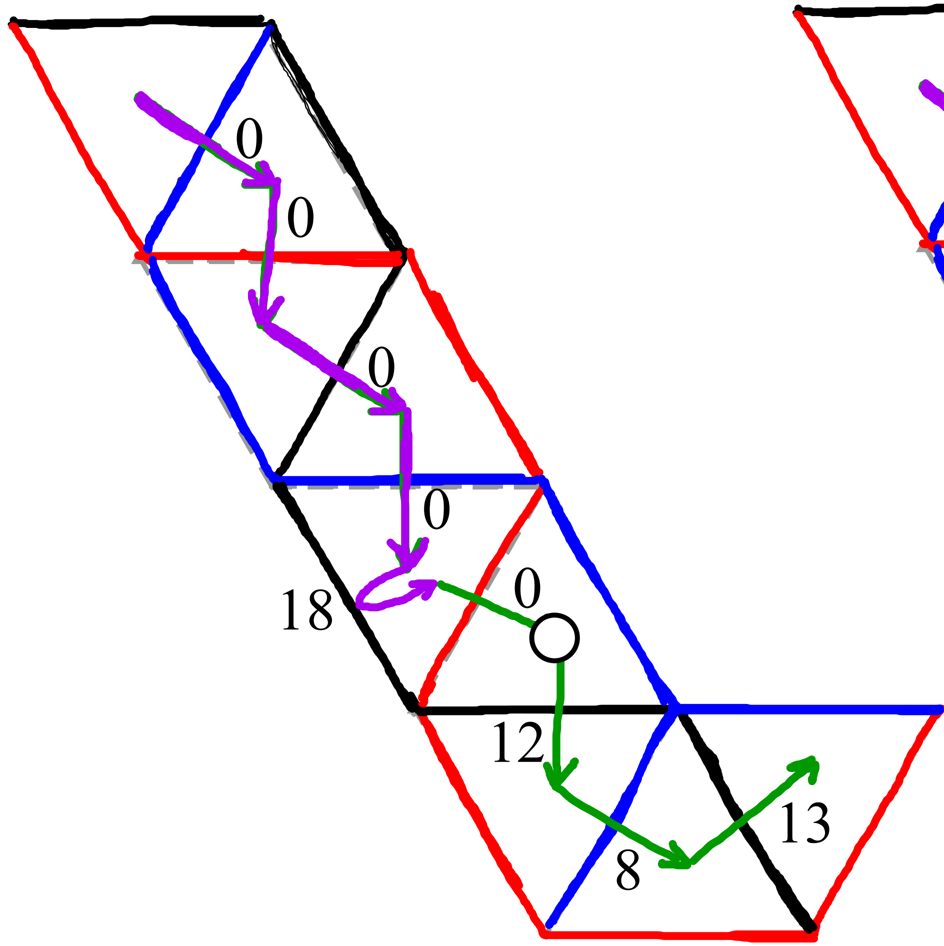
8
13
b6



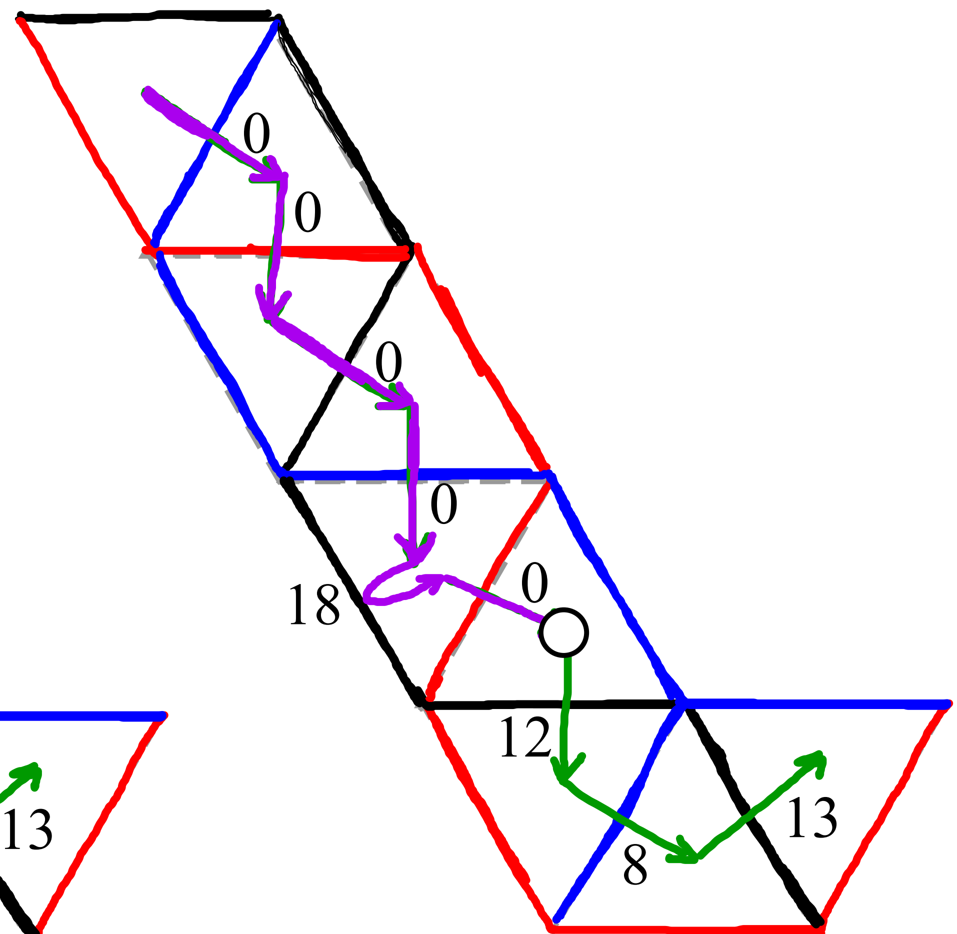
b_6



b'_6

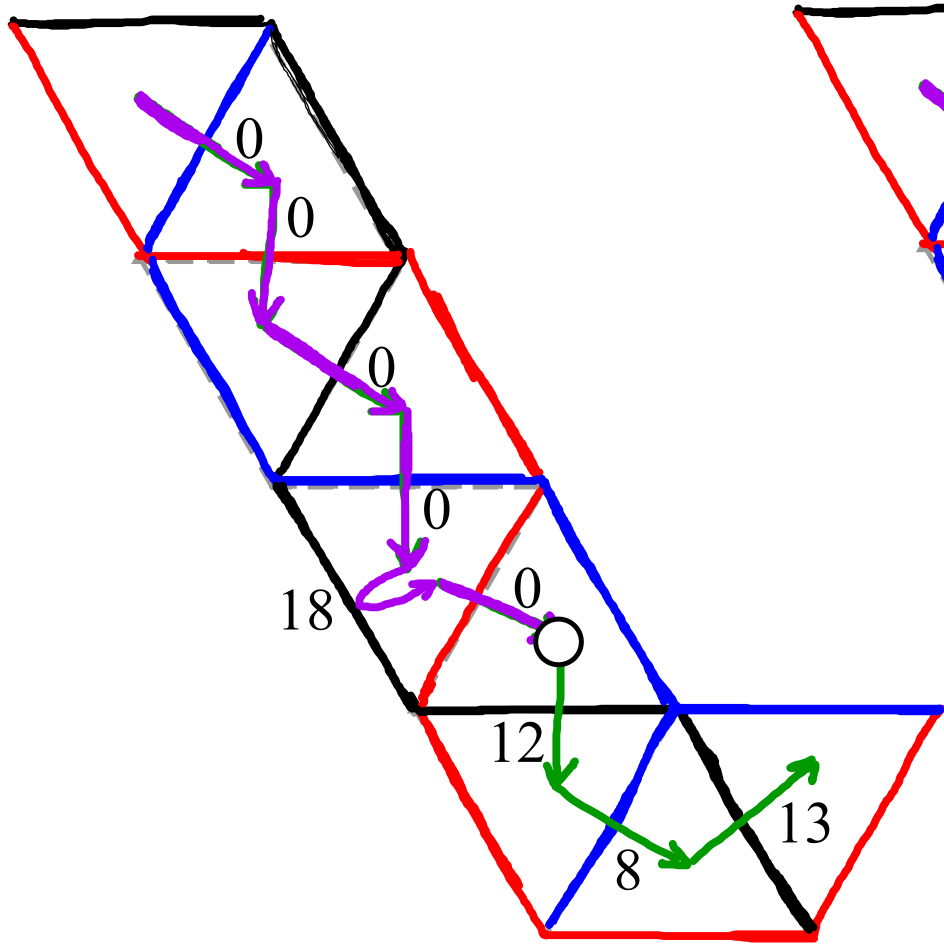


b'_6

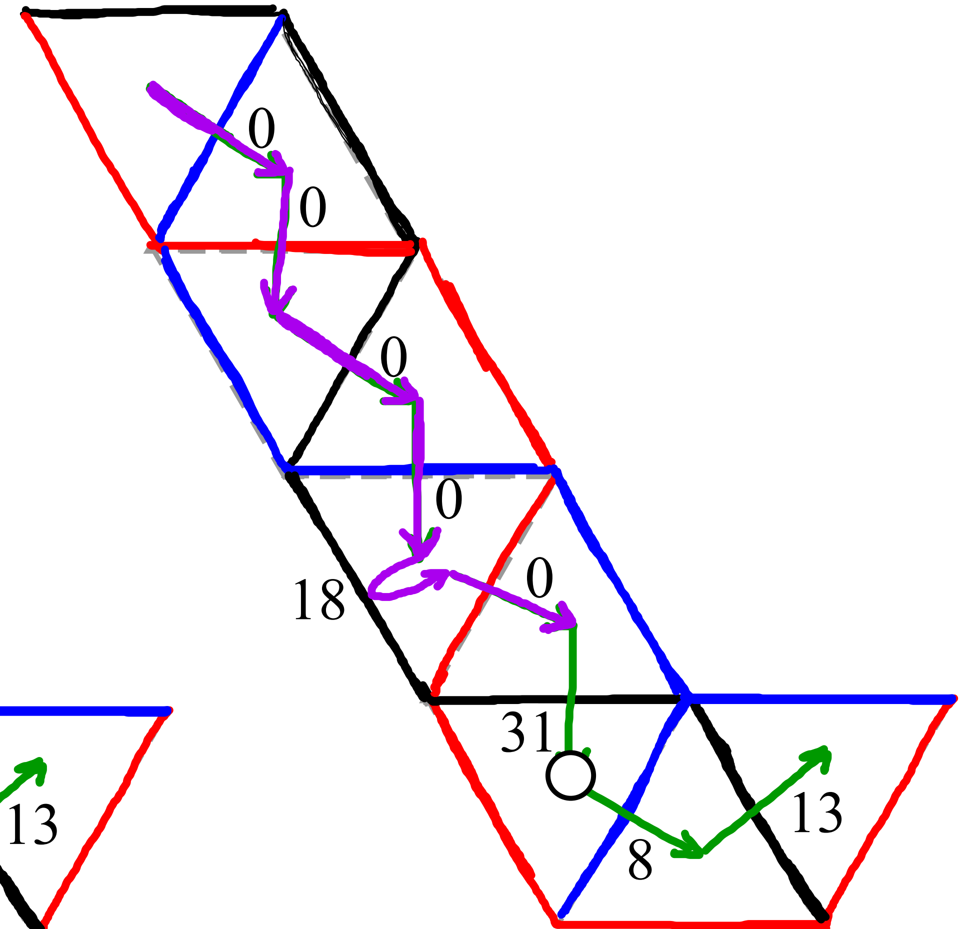


b_7

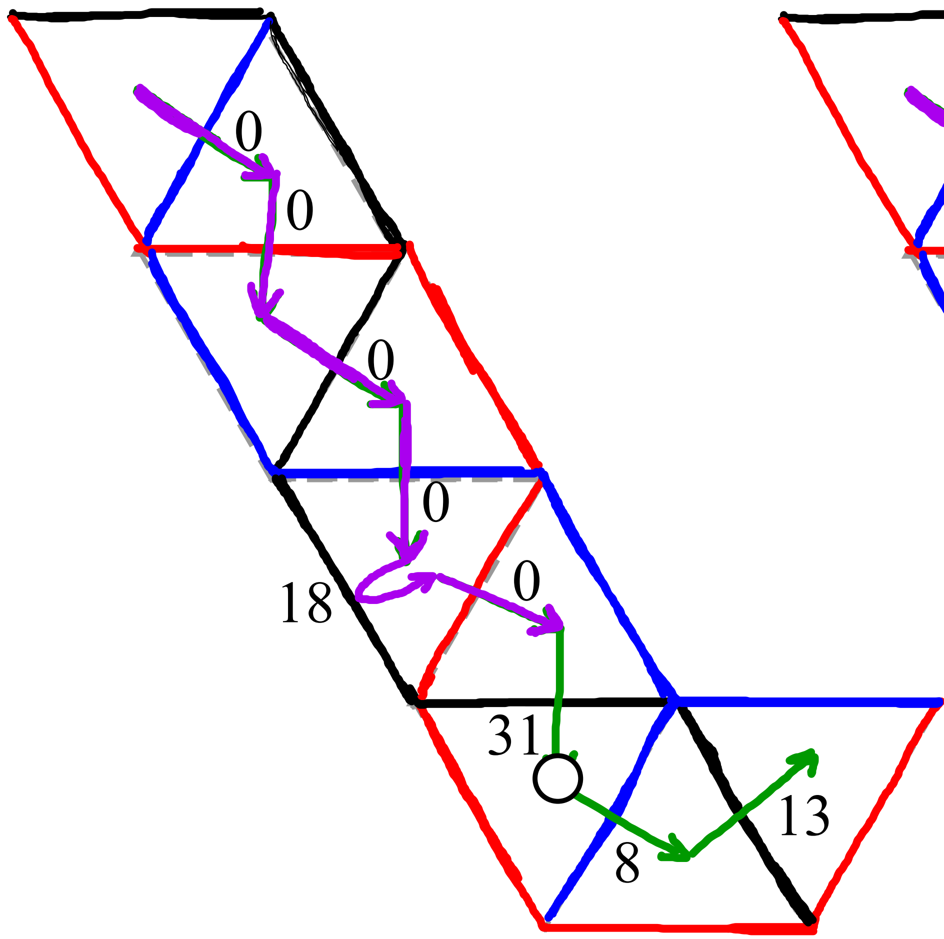
(18)(12)=31



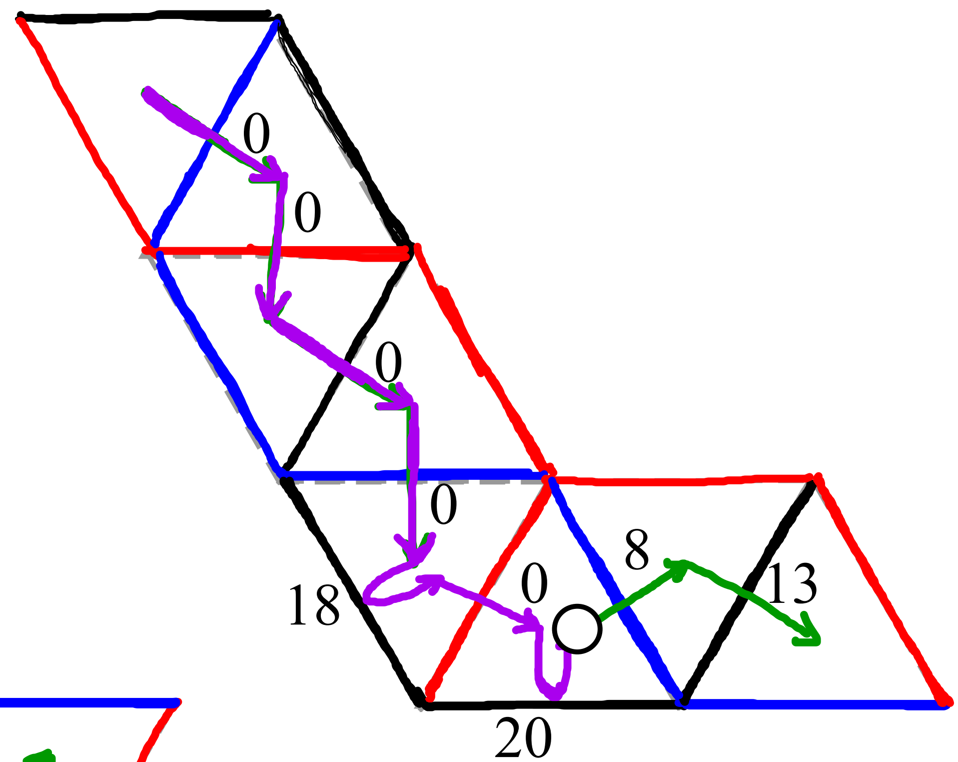
b_7



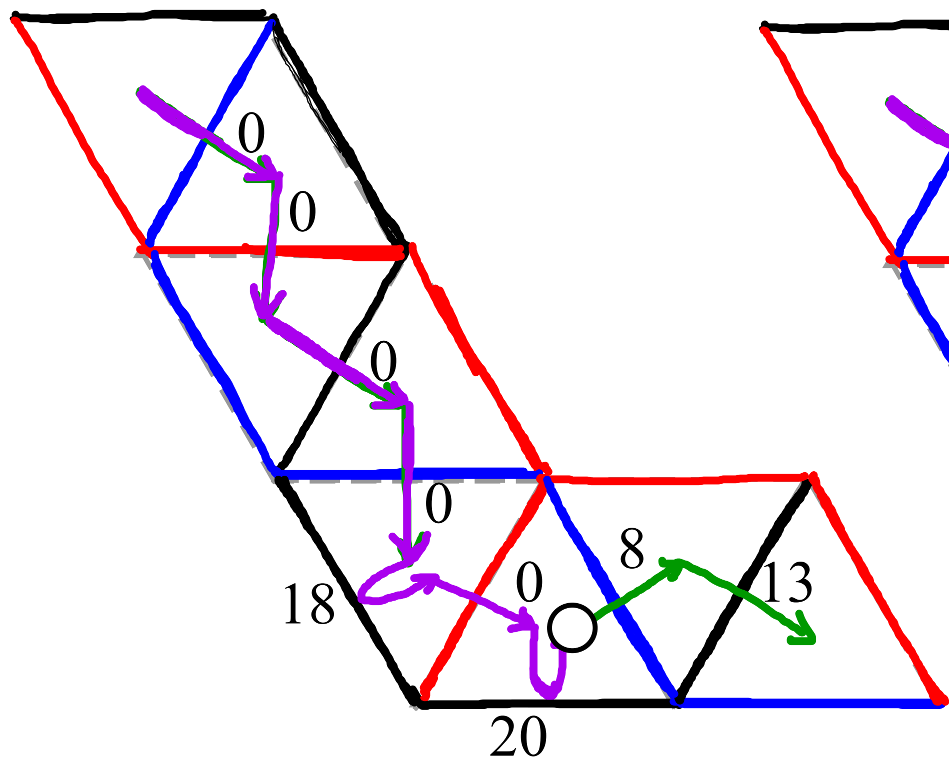
b'_7



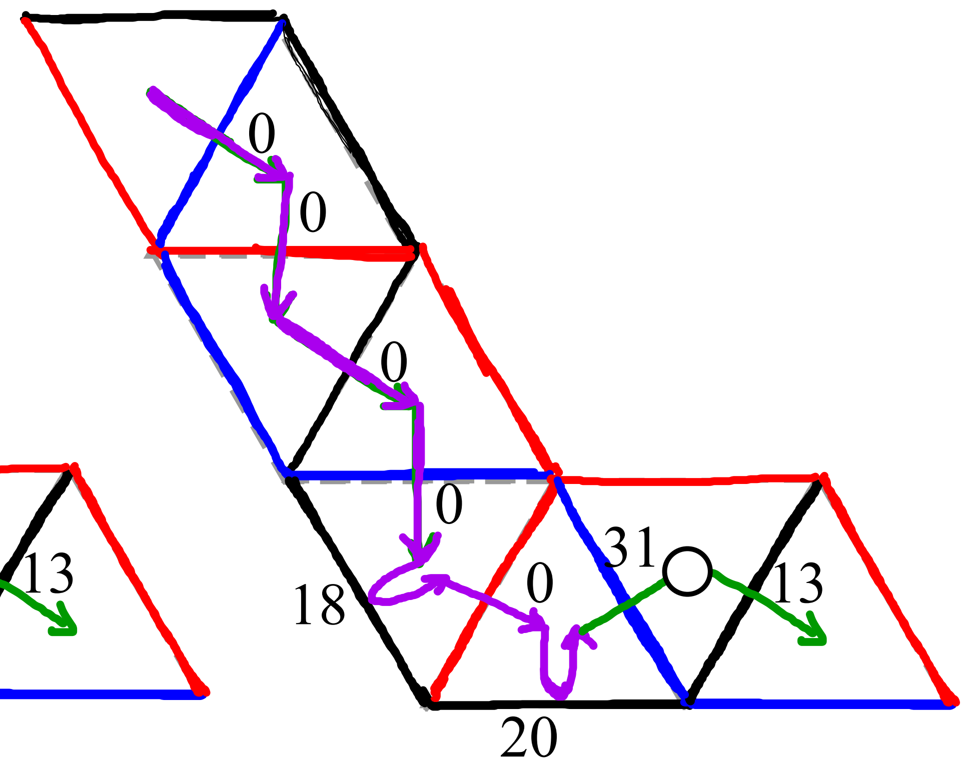
b'_7



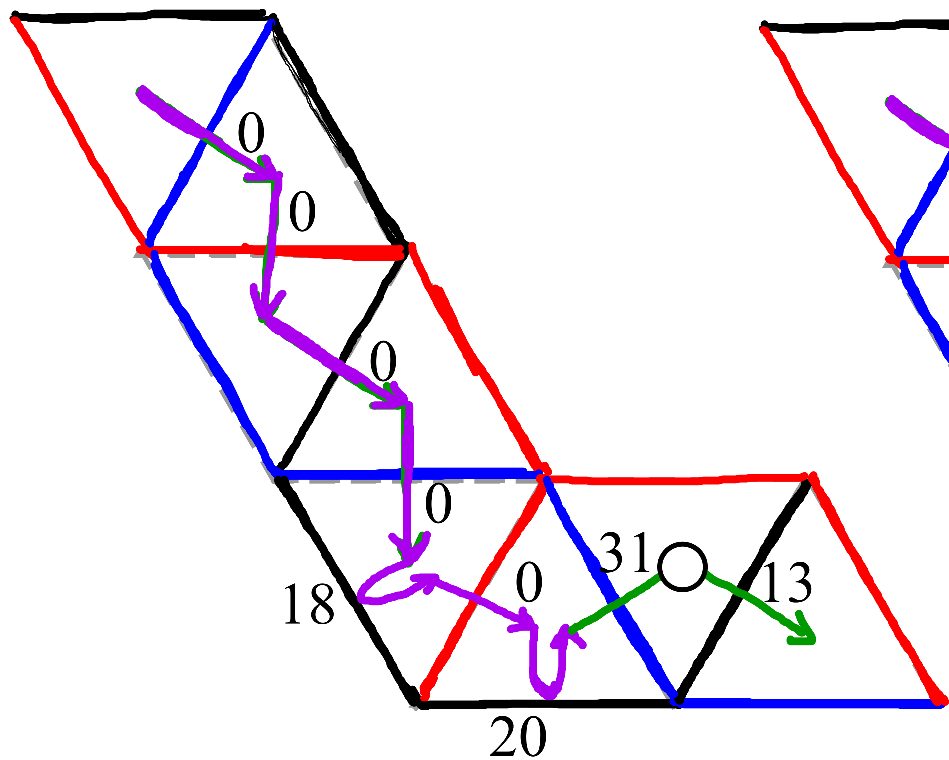
b_8



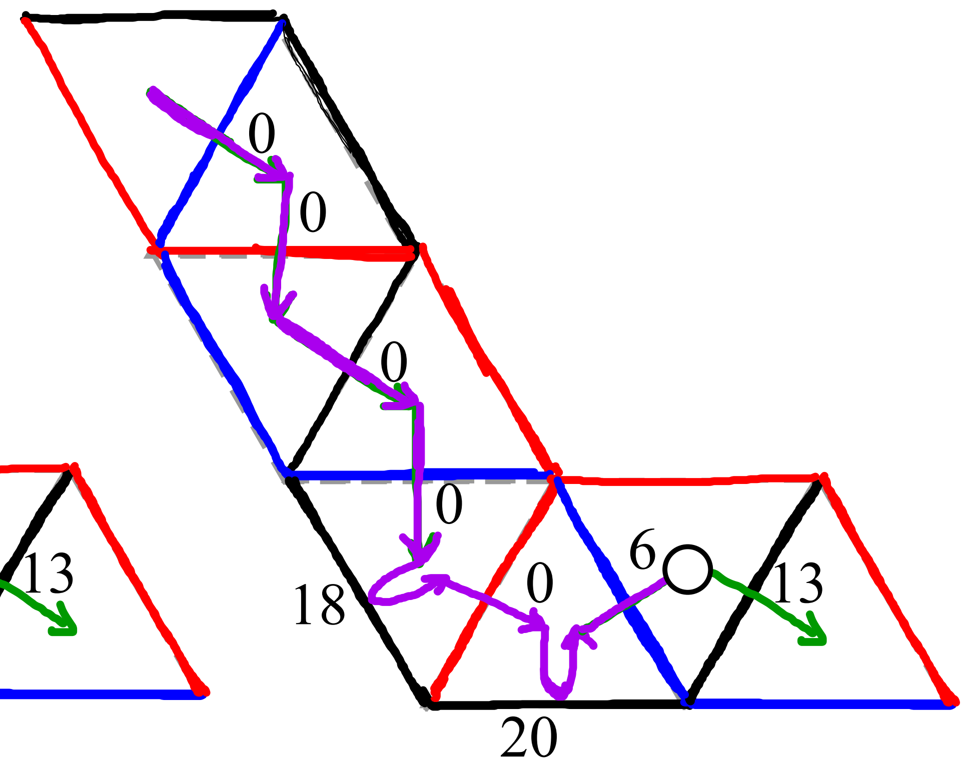
b_8



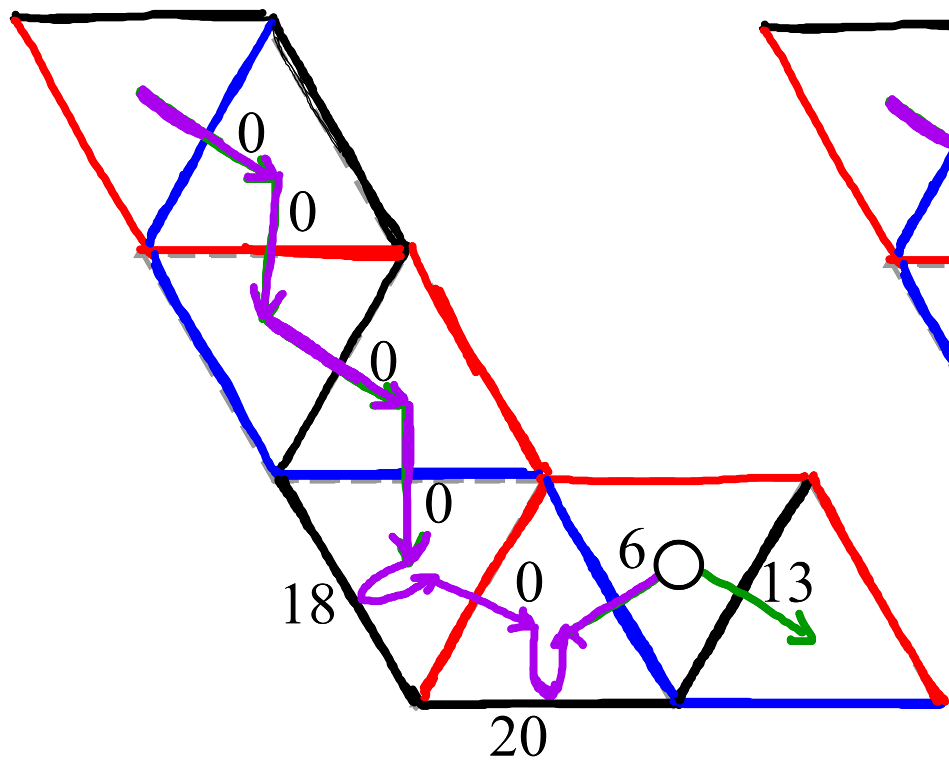
b'_8



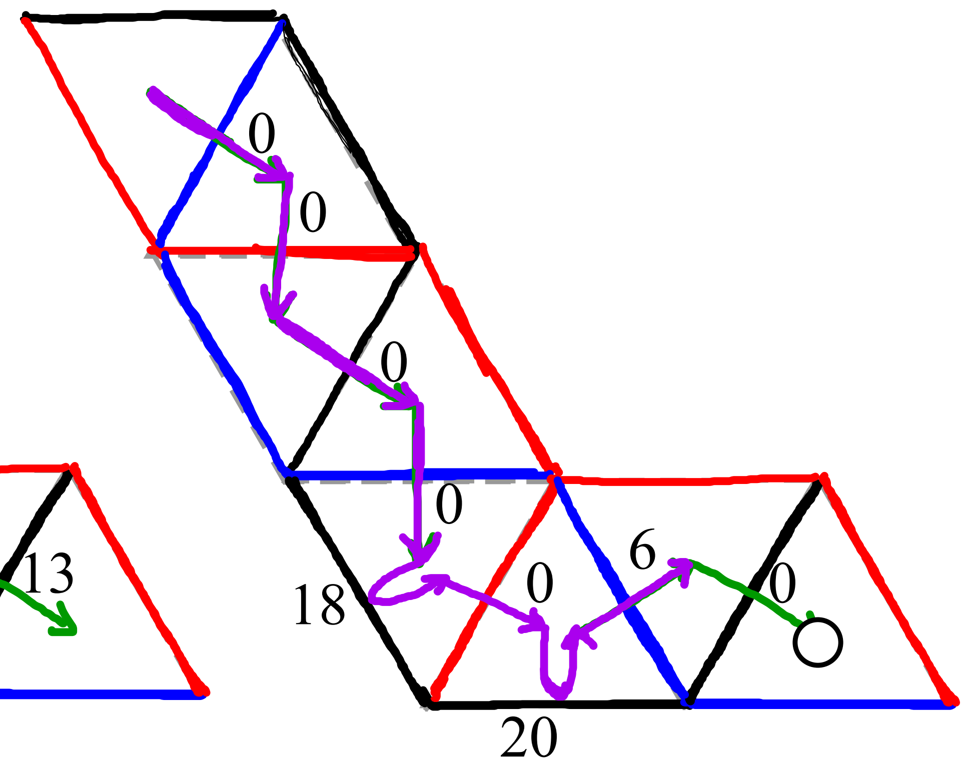
b'_8



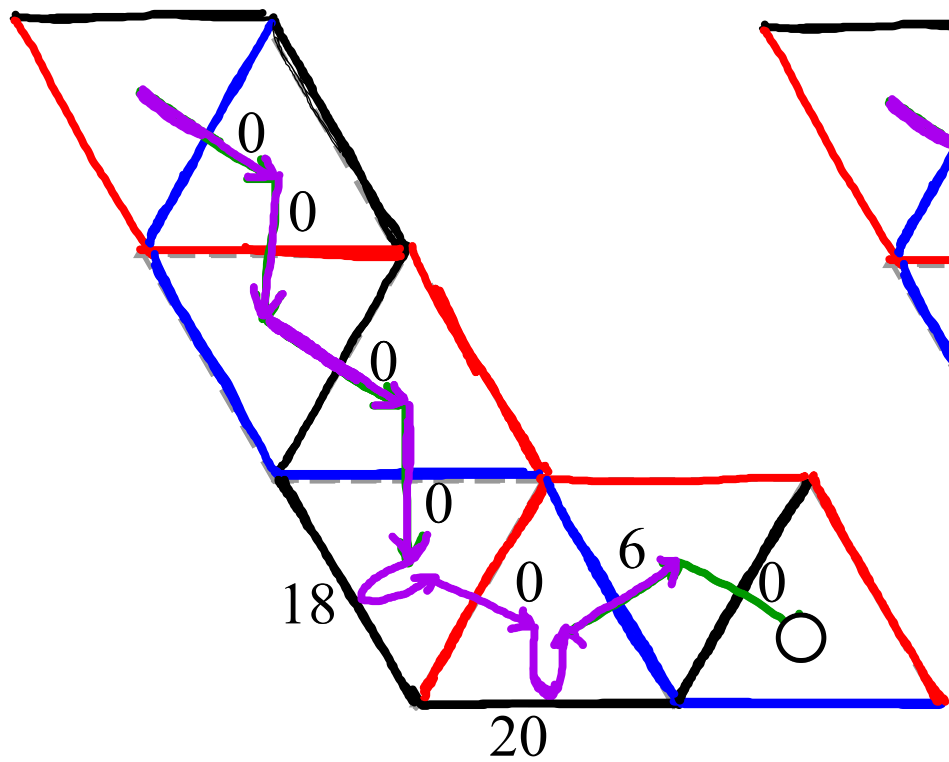
b_9

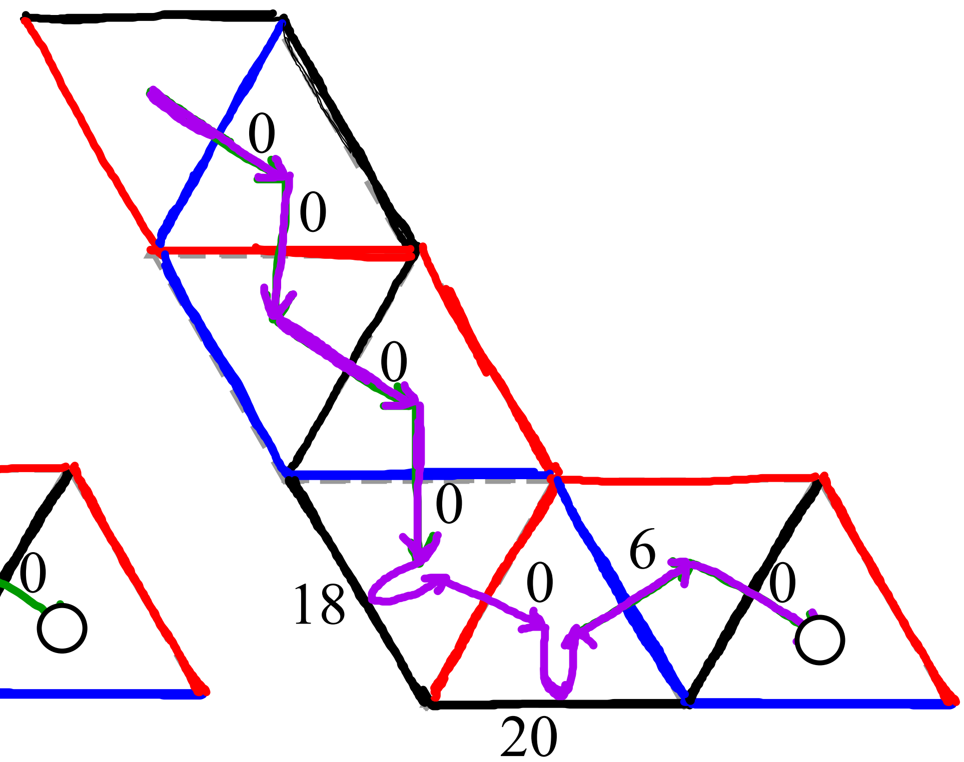


b_9



b'_9





$$b'_9 = b_{10}$$