

Study $H_T^*(G/B)$

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Examples Type C_n , $G = Sp_4$

$$W_0 = \left\{ \begin{array}{cc} & 1 \\ s_1 & s_2 \\ s_1 s_2 & s_2 s_1 \\ s_1 s_2 s_1 & s_2 s_1 s_2 \\ & s_1 s_2 s_1 s_2 \end{array} \right\} \quad \text{with } s_i^2 = 1 \text{ and } s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$$

acts on $\mathbb{C}[y_1, y_2]$ by

$$\begin{array}{ll} s_1 y_1 = y_2, & s_2 y_1 = y_1, \\ s_1 y_2 = y_1, & s_2 y_2 = -y_2. \end{array}$$

So $\mathbb{C}[y_1, y_2]^{W_0} = \mathbb{C}[y_1^2 + y_2^2, y_1^2 y_2^2]$

$$R^+ = \left\{ \begin{array}{l} y_{\alpha_1} = y_1 - y_2 \\ y_{\alpha_2} = 2y_2 \\ y_{\alpha_1 + \alpha_2} = y_1 + y_2 \\ y_{2\alpha_1 + \alpha_2} = 2y_1 + y_2 \end{array} \right\} \quad \text{and} \quad R^- = \left\{ \begin{array}{l} y_{-\alpha_1} = y_2 - y_1 \\ y_{-\alpha_2} = -2y_2 \\ y_{-\alpha_1 - \alpha_2} = -y_1 - y_2 \\ y_{-2\alpha_1 - \alpha_2} = -2y_1 \end{array} \right\}$$

The Borel model for $H_T^*(G/B)$

$$\begin{aligned} H_T^*(G/B) &= \frac{\mathbb{C}[x_1, x_2, y_1, y_2]}{\langle f(x_1, x_2) - f(y_1, y_2) \mid f \in \mathbb{C}[x_1, x_2]^{W_0} \rangle} \\ &= \frac{\mathbb{C}[x_1, x_2, y_1, y_2]}{\langle x_1^2 + x_2^2 = y_1^2 + y_2^2, x_1 x_2 = y_1 y_2 \rangle} \end{aligned}$$

The moment graph model

$$H_T^*(G/B) \xrightarrow{\cong} \bigoplus_{w \in W_0} \mathbb{C}[y_1, y_2]$$

$$f(x_1, x_2) \mapsto (wf(y_1, y_2))_{w \in W_0}$$

$$g(y_1, y_2) \mapsto (g(y_1, y_2))_{w \in W_0}$$

is an injective ring homomorphism

δ_0	y_1	y_2	y_1	y_2	x_1	x_2
$y_1 =$	y_1	y_1	y_2	y_2	y_2	y_1
	y_1	y_1	y_2	y_2	$-y_2$	y_2
	y_1	y_1	y_2	y_2	$-y_1$	$-y_2$
	y_1		y_2		$-y_1$	$-y_2$
			y_2			$-y_2$

$x_{-\alpha_1} = x_2 - x_1 =$	$y_2 - y_1$	$-y_2 - y_1$	$-2y_2$
	$y_1 - y_2$	$-y_1 - y_2$	$2y_2$
	$y_1 + y_2$	$-y_1 - y_2$	$-2y_1$
	$y_2 + y_1$	$-y_1 + y_2$	$2y_1$
	$-y_2 + y_1$		$-2y_2$
			$2y_2$

BGG-operators and Bott-Samelson classes

$A_i = (1 + t_{s_i}) \frac{1}{x_{-\alpha_i}}$ where $t_{s_i}(f_w)_{w \in W_0} = (f_{s_i z})_{z \in W_0}$

The Bott-Samelson classes are

$[z_{pt}] = \begin{cases} y_{R^-}, & \text{if } w=1 \\ 0, & \text{if } w \neq 1 \end{cases}$ where $y_{R^-} = \prod_{\alpha \in R^+} y_{-\alpha}$

and, for a sequence $s_i \dots s_{i_2}$ (not necessarily reduced)

$$[z_{i_1 \dots i_2}] = A_{i_1} \dots A_{i_2} [z_{pt}]$$

For example,

$$[z_{21}] = A_1 [z_{21}] = (1+t_{s_1}) \frac{1}{x_{-a_1}} [z_{21}]$$

$$= (1+t_{s_1}) \begin{pmatrix} \frac{1}{y_2-y_1} & \frac{1}{-y_2-y_1} \\ \frac{1}{y_1+y_2} & \frac{1}{-y_1-y_2} \\ \frac{1}{-y_2+y_1} & 0 \end{pmatrix} \begin{pmatrix} (-y_1-y_2)(-2y_1) & (-y_1-y_2)(-2y_1) \\ (-y_1-y_2)(-2y_2) & (-y_1-y_2)(-2y_2) \\ 0 & (-y_1-y_2)(-2y_2) \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= (1+t_{s_1}) \begin{pmatrix} \frac{-y_1-y_2}{y_2-y_1} (-2y_1) & -2y_1 \\ \frac{-y_1-y_2}{y_1-y_2} (-2y_2) & -2y_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-y_1-y_2}{y_2-y_1} (-2y_1) & -2y_1 \\ \frac{-y_1-y_2}{y_1-y_2} (-2y_2) & -2y_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{-y_1-y_2}{y_1-y_2} (-2y_2) & 0 \\ \frac{-y_1-y_2}{y_2-y_1} (-2y_1) & 0 \\ -2y_1 & 0 \\ -2y_2 & 0 \end{pmatrix} = \begin{pmatrix} (-2)(-y_1-y_2) & -2y_1 \\ (-2)(-y_1-y_2) & -2y_2 \\ -2y_1 & -2y_2 \\ -2y_2 & 0 \end{pmatrix}$$

