Generalized Schubert Calculus

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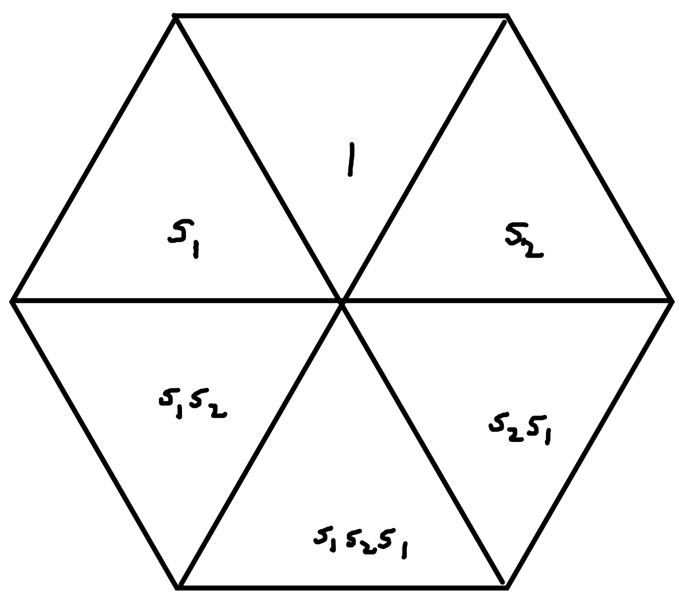
$H_T(G/B) = (S \otimes S) \cdot 1$

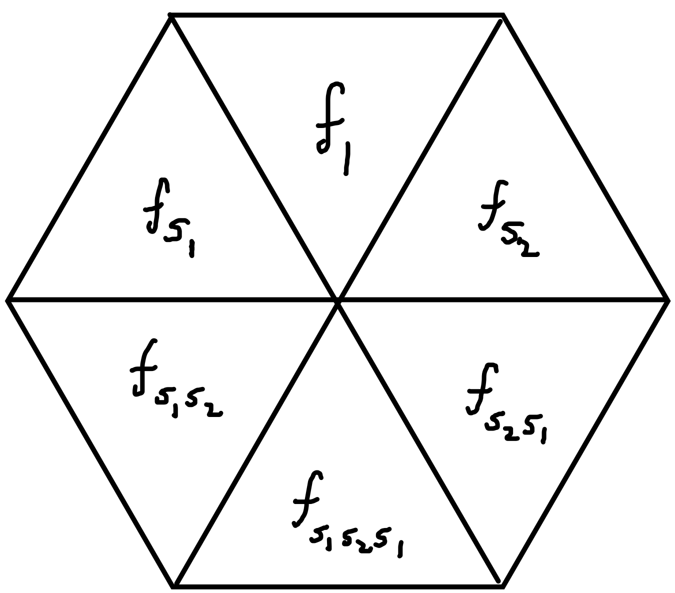
$W_0 = \langle s_1, s_2 \mid s_i^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$ acts on

$H_T(pt) = S = \mathbb{C}[y_1, y_2, y_3]$ by permuting y_1, y_2, y_3

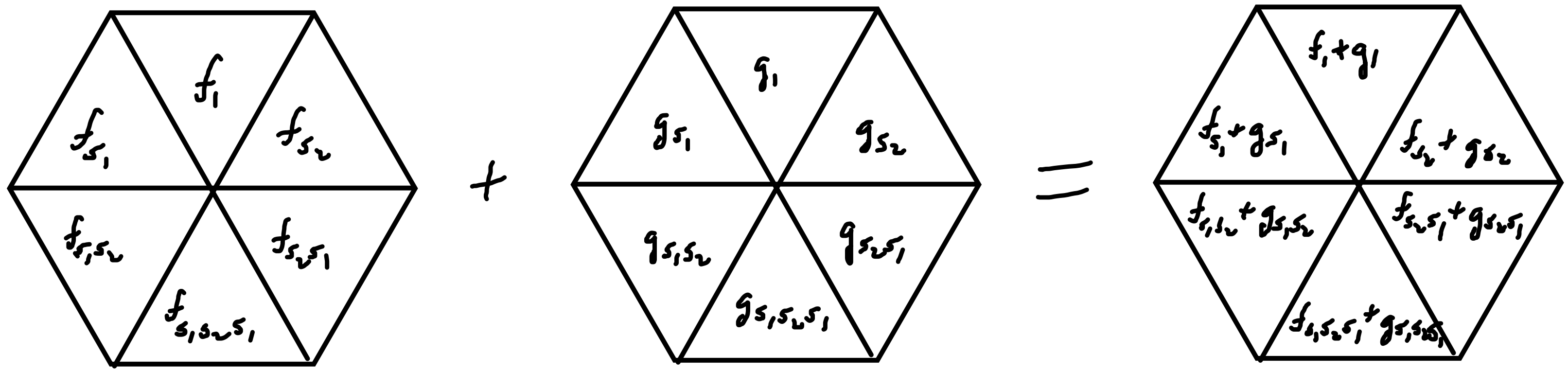


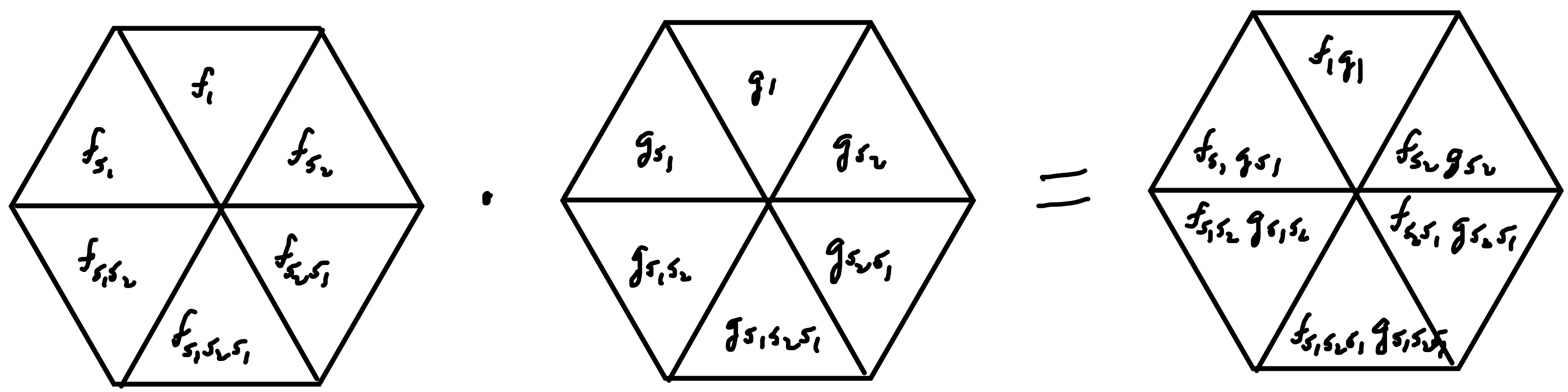
Put a polynomial $f_w \in \mathbb{C}[y_1,y_2,y_3]$ in each chamber.

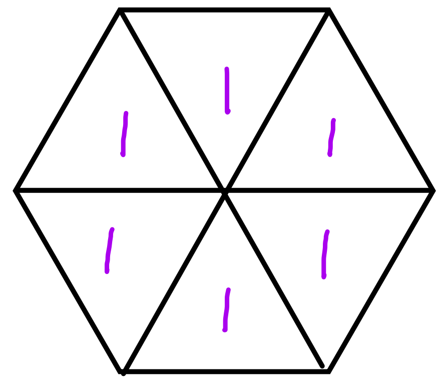
- addition and multiplication are pointwise
- $S \otimes S = \mathbb{C}[x_1,x_2,x_3,y_1,y_2,y_3]$ acts on $\displaystyle\bigoplus_{w \in W_0} \mathbb{C}[y_1,y_2,y_3]$
- $\mathbb{C}[W_0] = \mathrm{span}\{t_w \mid w \in W_0\}$ acts on



$$\in \bigoplus_{w \in W_0} \mathbb{C}[y_1,y_2,y_3]$$


f_1
f_{s_1}
f_{s_2}
$f_{s_1s_2}$
$f_{s_2s_1}$
$f_{s_1s_2s_1}$
+
g_1
g_{s_1}
g_{s_2}
$g_{s_1s_2}$
$g_{s_2s_1}$
$g_{s_1s_2s_1}$
=
f_1+g_1
$f_{s_1}+g_{s_1}$
$f_{s_2}+g_{s_2}$
$f_{s_1s_2}+g_{s_1s_2}$
$f_{s_2s_1}+g_{s_2s_1}$
$f_{s_1s_2s_1}+g_{s_1s_2s_1}$


f_1
f_{s_1}
f_{s_2}
$f_{s_1s_2}$
$f_{s_2s_1}$
$f_{s_1s_2s_1}$
·
g_1
g_{s_1}
g_{s_2}
$g_{s_1s_2}$
$g_{s_2s_1}$
$g_{s_1s_2s_1}$
=
f_1g_1
$f_{s_1}g_{s_1}$
$f_{s_2}g_{s_2}$
$f_{s_1s_2}g_{s_1s_2}$
$f_{s_2s_1}g_{s_2s_1}$
$f_{s_1s_2s_1}g_{s_1s_2s_1}$

$1 =$ 

$$y_{-\alpha_1} = y_2 - y_1 \qquad x_{-\alpha_1} = x_2 - x_1$$

$$y_{-\alpha_2} = y_3 - y_2 \qquad x_{-\alpha_2} = x_3 - x_2$$

$$y_{-(\alpha_1+\alpha_2)} = y_3 - y_1 \qquad x_{-(\alpha_1+\alpha_2)} = x_3 - x_1$$

Put a polynomial $f_w \in \mathbb{C}[y_1,y_2,y_3]$ in each chamber.

- addition and multiplication are pointwise
- $S \otimes S = \mathbb{C}[x_1,x_2,x_3,y_1,y_2,y_3]$ acts on $\displaystyle\bigoplus_{w\in W_0} \mathbb{C}[y_1,y_2,y_3]$
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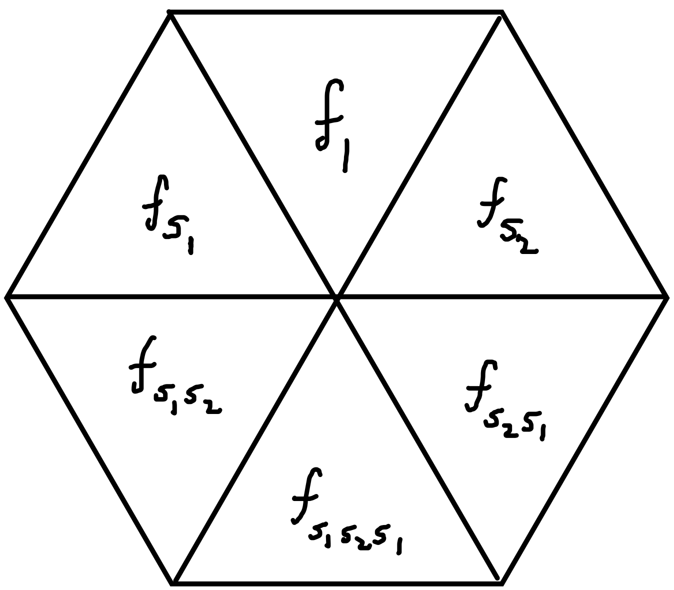
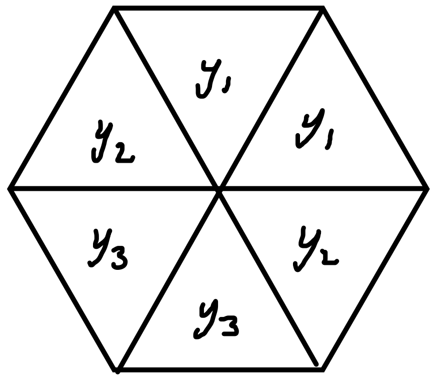
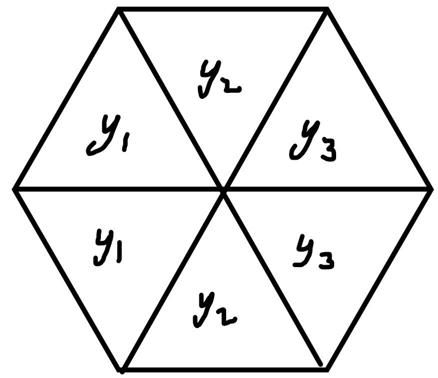


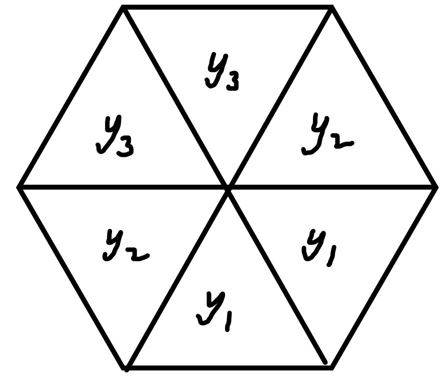
$\in \displaystyle\bigoplus_{w\in W_0} \mathbb{C}[y_1,y_2,y_3]$



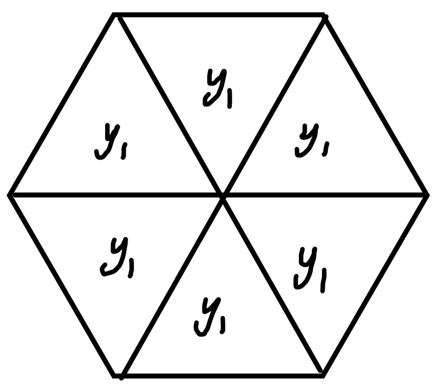
x_1



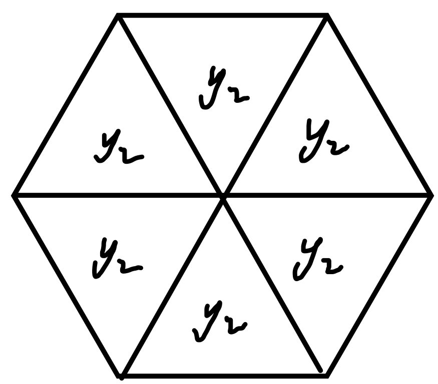
x_2



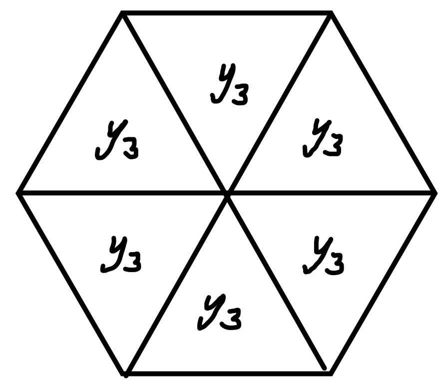
x_3



y_1

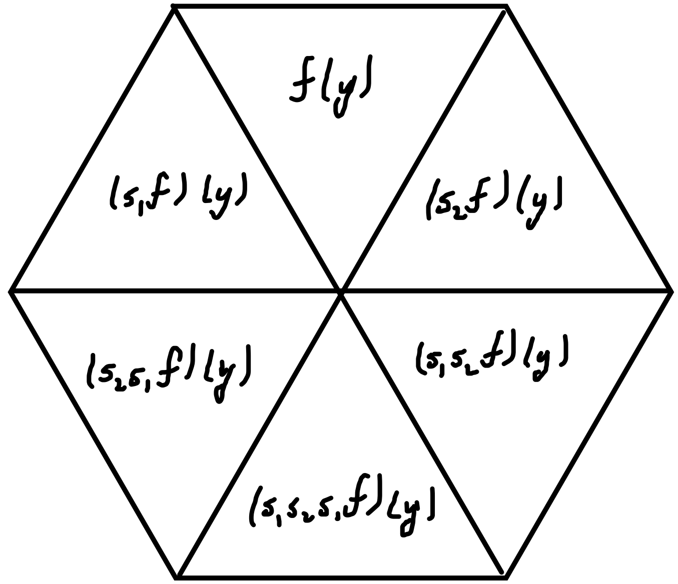


y_2

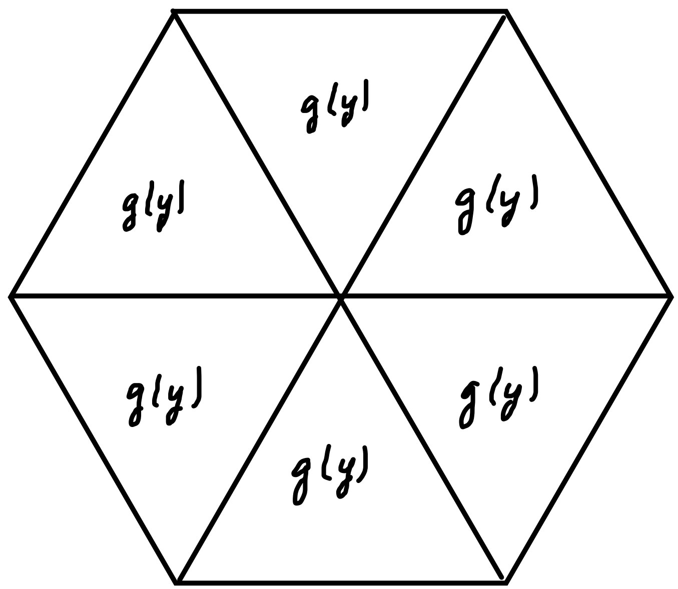


y_3

$$f(x_1, x_2, x_3) = f(x) =$$



$$g(y_1, y_2, y_3) = g(y) =$$



Put a polynomial $f_w \in \mathbb{C}[y_1,y_2,y_3]$ in each chamber.

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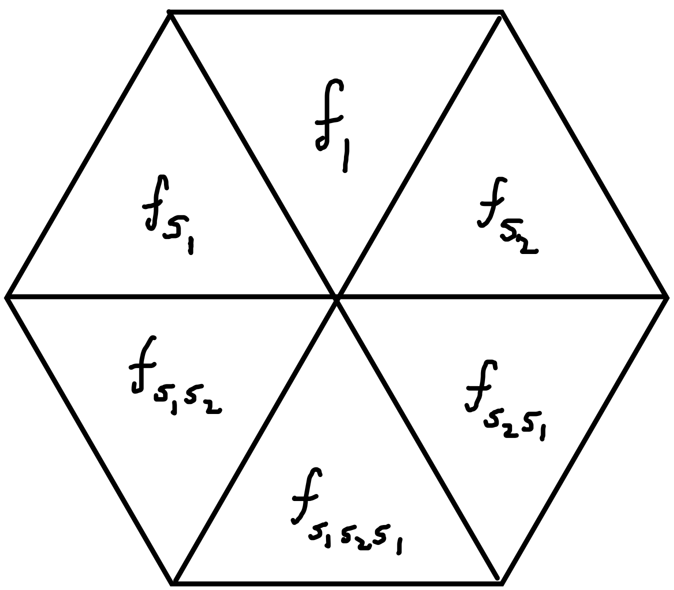
- $\mathbb{C}[W_0] = \mathrm{span}\{t_w \mid w \in W_0\}$ acts on

$$\bigoplus_{w \in W_0} \mathbb{C}[y_1,y_2,y_3]$$



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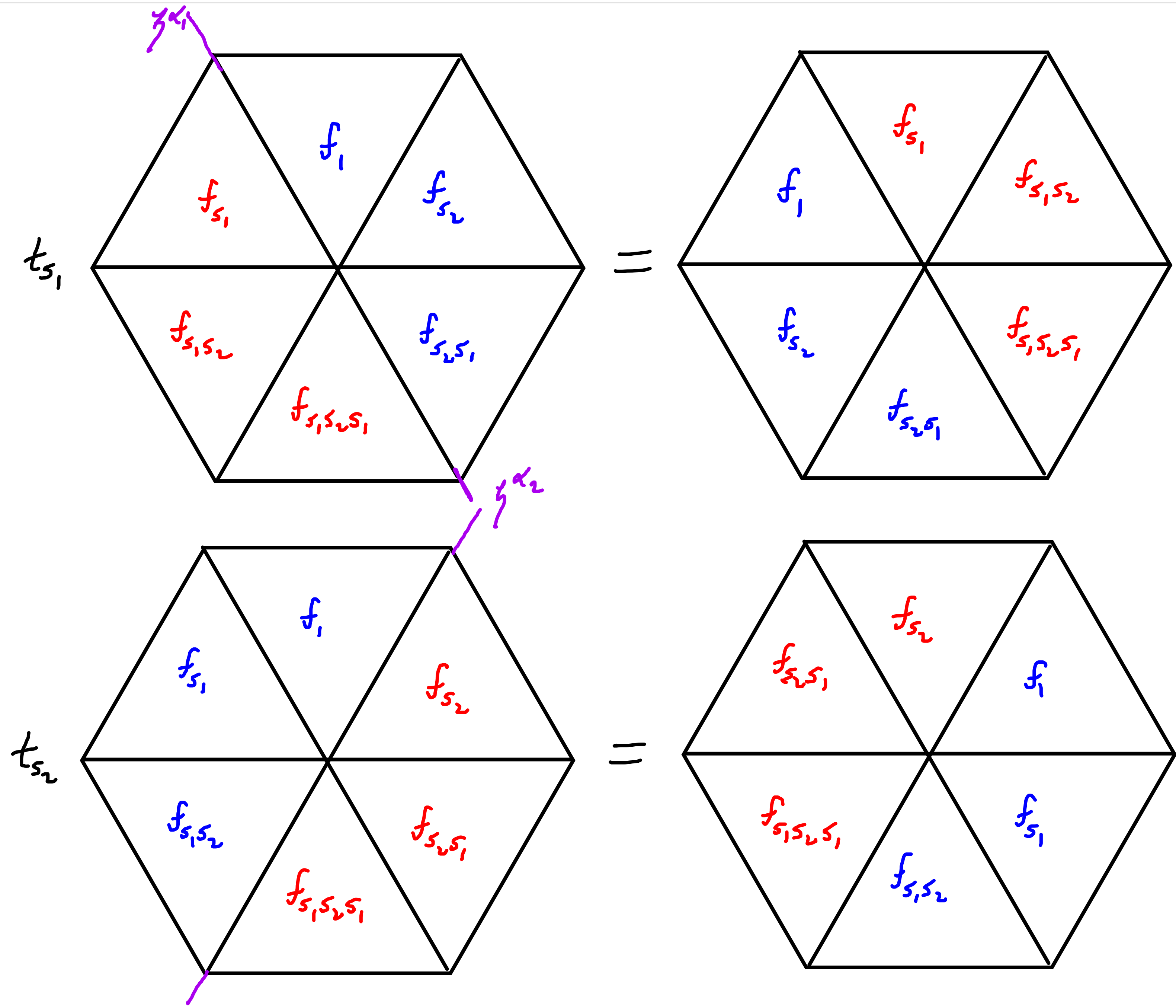
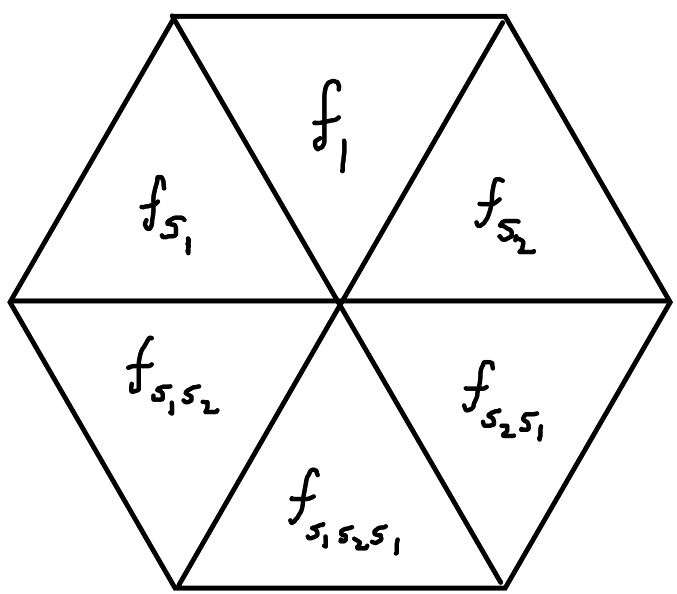

t_{s_1}
ζ^{α_1}
f_1
f_{s_1}
f_{s_2}
$f_{s_1 s_2}$
$f_{s_2 s_1}$
$f_{s_1 s_2 s_1}$
=
f_{s_1}
f_1
$f_{s_1 s_2}$
f_{s_2}
$f_{s_1 s_2 s_1}$
$f_{s_2 s_1}$
ζ^{α_2}
t_{s_2}
f_1
f_{s_1}
f_{s_2}
$f_{s_1 s_2}$
$f_{s_2 s_1}$
$f_{s_1 s_2 s_1}$
=
f_{s_2}
$f_{s_2 s_1}$
f_1
$f_{s_1 s_2 s_1}$
f_{s_1}
$f_{s_1 s_2}$

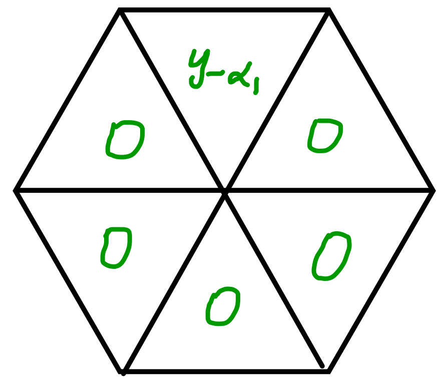
Put an element $f_w \in S$ in each chamber.

- addition and multiplication are pointwise
- $S \otimes S$ acts on
- $\mathbb{C}[W_0] = \mathrm{span}\{t_w \mid w \in W_0\}$ acts on $\bigoplus_{w \in W_0} S$

$$H_T(G/B) = (S \otimes S) \cdot 1$$



$$\in \bigoplus_{w \in W_0} S$$



is an element of $\bigoplus_{w \in W_0} S$

that is not an element of

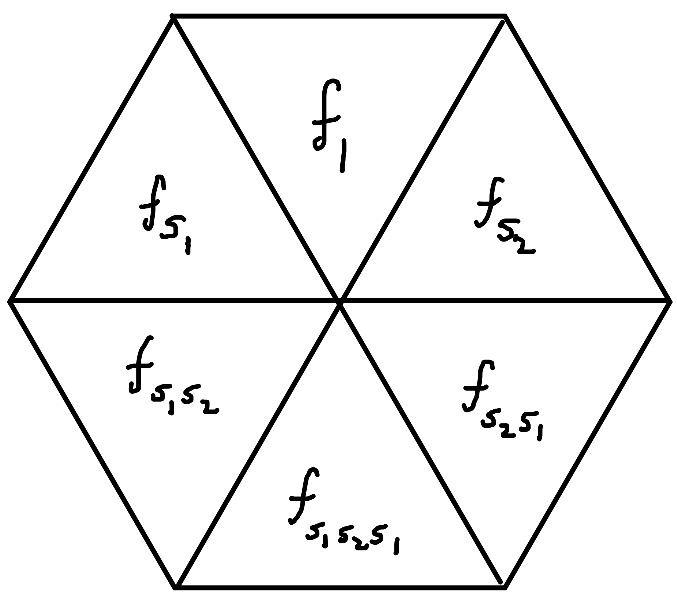
$$H_T(G/B) = (S \otimes S) \cdot 1 \subsetneq \bigoplus_{w \in W_0} S$$

Put an element $f_w \in S$ in each chamber.

- addition and multiplication are pointwise
- $S \otimes S$ acts on $\bigoplus_{w \in W_0} S$
- $\mathbb{C}[W_0] = \mathrm{span}\{t_w \mid w \in W_0\}$ acts on $\bigoplus_{w \in W_0} S$

$$H_T(G/B) = (S \otimes S) \cdot 1$$

$$\in \bigoplus_{w \in W_0} S$$



$H_T(G/B) = (S \otimes S)\cdot 1$ has a basis (over $S = 1 \otimes S$) of

Schubert classes $\{[X_w] \mid w \in W_0\}$

BGG/Demazure

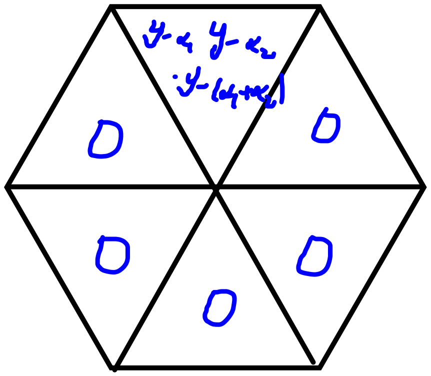
Let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w.

$$[X_w] = A_{i_1} \cdots A_{i_\ell} \cdot [X_{pt}]$$

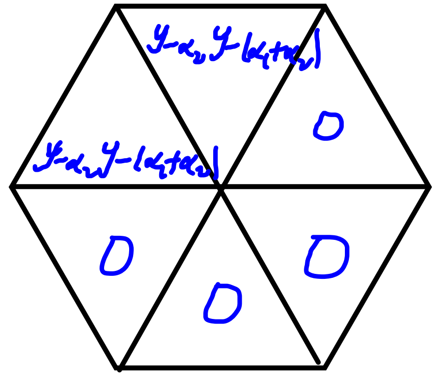
where $A_i = (1 + t_{s_i}) \frac{1}{x_{-\alpha_i}}$

$x_{-\alpha_1} = x_2 - x_1$

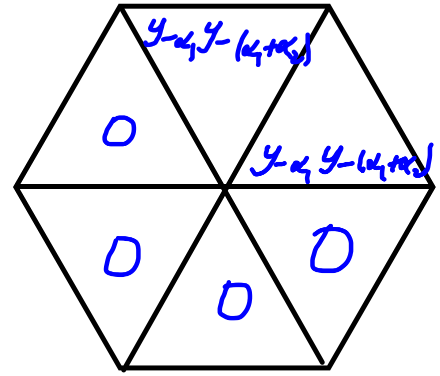
$x_{-\alpha_2} = x_3 - x_2$



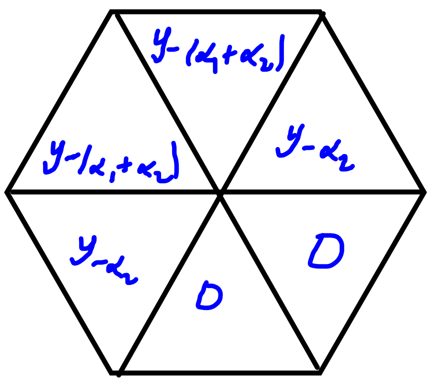
$[X_{pt}]$



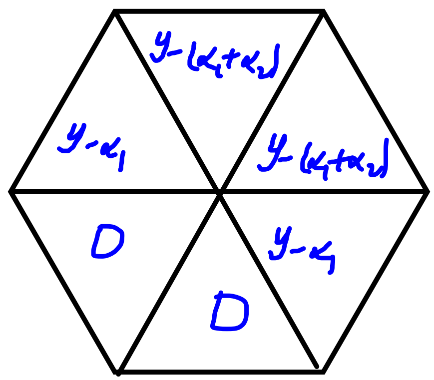
$[X_{s_1}]$



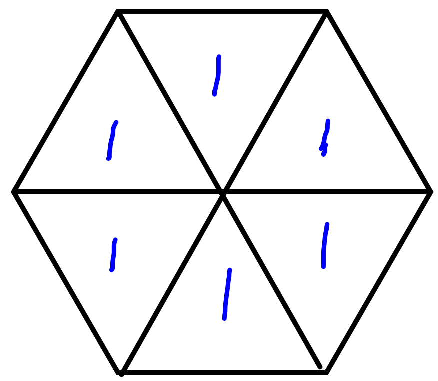
$[X_{s_2}]$



$[X_{s_1 s_2}]$

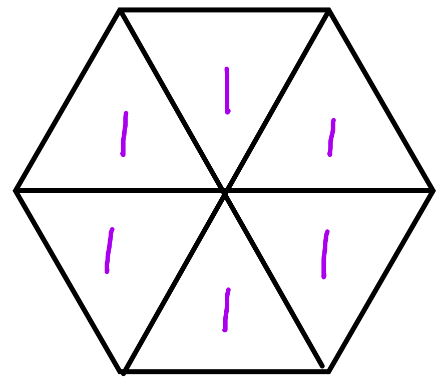


$[X_{s_2 s_1}]$



$[X_{s_1 s_2 s_1}]$

Schubert Classes

$$1 = $$ 

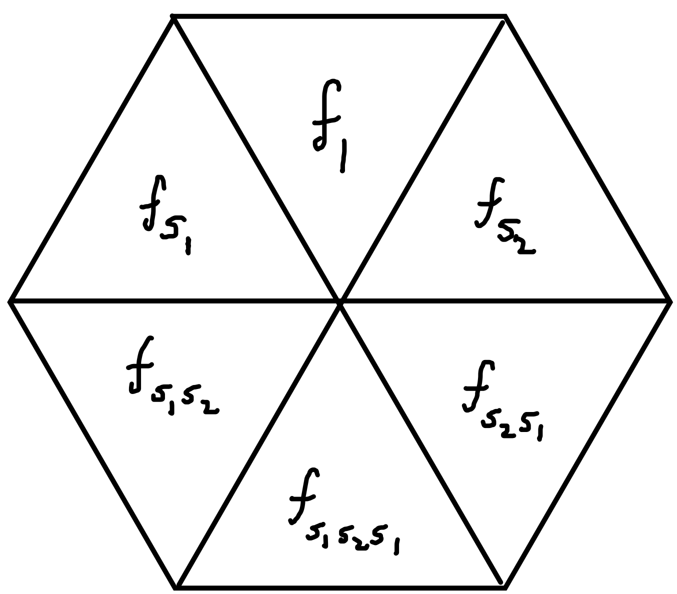
$y_{-\alpha_1} = y_2 - y_1$ $\qquad$ $x_{-\alpha_1} = x_2 - x_1$

$y_{-\alpha_2} = y_3 - y_2$ $\qquad$ $x_{-\alpha_2} = x_3 - x_2$

$y_{-(\alpha_1+\alpha_2)} = y_3 - y_1$ $\qquad$ $x_{-(\alpha_1+\alpha_2)} = x_3 - x_1$

$$H_T(G/B) = (S \otimes S) \cdot 1 \qquad \text{in} \quad \bigoplus_{w \in W_0} S$$

$$S = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu$$



$$\in \quad \bigoplus_{w \in W_0} S$$

Ordinary cohomology $S = H_T(pt)$

$$H_T(pt) = S(\mathfrak{h}_{\mathbb{Z}}^*) = \mathbb{C}[y_1, \dots, y_n] = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]$$

with

$$y_{\lambda+\mu} = y_\lambda + y_\mu$$

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_3$$

$y_1 = y_{\varepsilon_1}$

$y_2 = y_{\varepsilon_2}$ $\qquad$ $y_\lambda = y_{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n} = \underbrace{y_{\varepsilon_1} + \cdots + y_{\varepsilon_1}}_{\lambda_1} + \cdots + \underbrace{y_{\varepsilon_n} + \cdots + y_{\varepsilon_n}}_{\lambda_n}$

$y_3 = y_{\varepsilon_3}$

<u>Ordinary cohomology</u> $S = H_T(pt)$

$$H_T(pt) = S(\mathfrak{h}_{\mathbb{Z}}^*) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]$$

with

$$y_{\lambda+\mu} = y_\lambda + y_\mu$$

<u>K-theory</u> $S = K_T(pt)$

$K_T(pt) = \mathbb{C}[y^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]$ with $y^\lambda y^\mu = y^{\lambda+\mu}$

$K_T(pt) = \mathbb{C}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}]$ with $y^\lambda = y^{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n}$

$$= (y^{\varepsilon_1})^{\lambda_1} \cdots (y^{\varepsilon_n})^{\lambda_n}$$

$y_1 = y^{\varepsilon_1}, \ y_2 = y^{\varepsilon_2}, \ y_3 = y^{\varepsilon_3}$

$$= y_1^{\lambda_1} \cdots y_n^{\lambda_n}$$

__Ordinary cohomology__ $S = H_T(pt)$

$$H_T(pt) = S(\mathfrak{h}^*_{\mathbb{Z}}) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}]$$

with

$$y_{\lambda+\mu} = y_\lambda + y_\mu$$

__K-theory__ $S = K_T(pt)$

$$K_T(pt) = \mathbb{C}[Y^\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}] \quad \text{with} \quad Y^\lambda Y^\mu = Y^{\lambda+\mu}$$

$$K_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}] \quad \text{with}$$

$$y_\lambda = 1 - Y^\lambda \qquad \text{so that} \qquad y_{\lambda+\mu} = y_\lambda + y_\mu - y_\lambda y_\mu$$

Ordinary cohomology $S = H_T(pt)$

$$H_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu$$

K-theory $S = K_T(pt)$

$$K_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu - y_\lambda y_\mu$$

Elliptic cohomology $S = Ell_T(pt)$

$$Ell_T(pt) = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}]] \quad \text{with}$$

$$y_{\lambda+\mu} = y_\lambda + y_\mu - a_1 y_\lambda y_\mu - a_2 y_\lambda^2 y_\mu - a_2 y_\mu y_\lambda^2 - 2a_3 y_\lambda^3 y_\mu$$
$$- 2a_3 y_\lambda y_\mu^3 + (a_1 a_2 - 3a_3) y_\lambda^2 y_\mu^2 + \cdots$$

where the elliptic curve is

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

Ordinary cohomology $S = H_T(pt)$

$$H_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}_\mathbb{Z}^*] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu$$

K-theory $S = K_T(pt)$

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Elliptic cohomology $S = Ell_T(pt)$

$$Ell_T(pt) = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}_\mathbb{Z}^*]] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu - a_1 y_\lambda y_\mu - \cdots$$

Cobordism $S = \Omega_T(pt)$

$$\Omega_T(pt) = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}_\mathbb{Z}^*]] \quad \text{with}$$

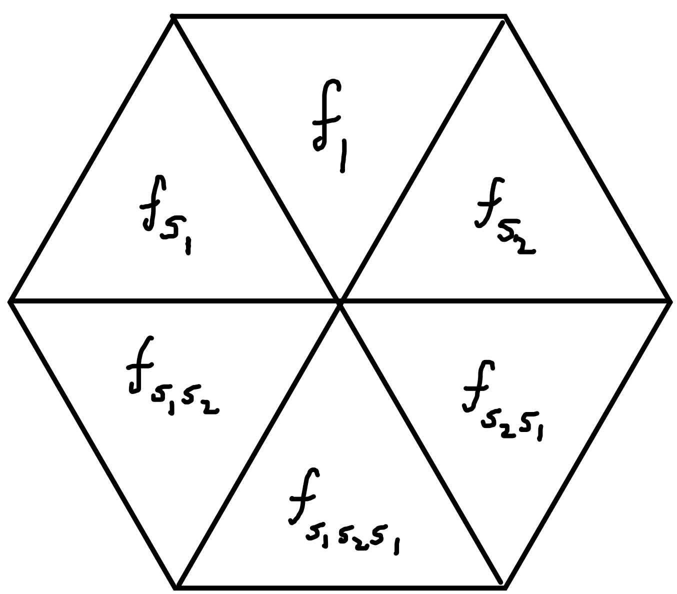
$$y_{\lambda+\mu} = y_\lambda + y_\mu + a_{11} y_\lambda y_\mu + a_{21} y_\lambda^2 y_\mu + a_{12} y_\lambda y_\mu^2 + a_{31} y_\lambda^3 y_\mu + a_{22} y_\lambda^2 y_\mu^2 + \cdots$$

where a_{ij} satisfy relations so that

$$y_{\lambda+\mu} = y_{\mu+\lambda}, \quad y_{(\lambda+\mu)+\nu} = y_{\lambda+(\mu+\nu)}, \quad y_{\lambda+(-\lambda)} = y_0 = 0$$

$$H_T(G/B) = (S \otimes S) \cdot 1 \qquad \text{in} \quad \bigoplus_{w \in W_0} S$$

$$H_T(pt) = S = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu$$



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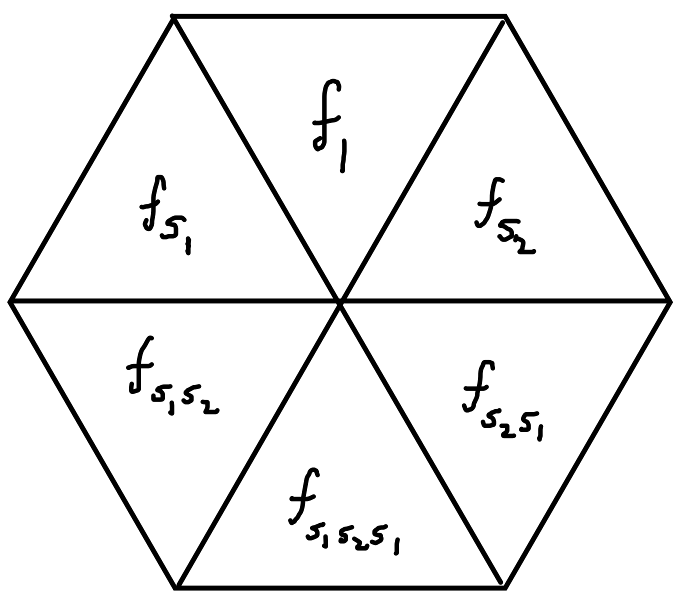
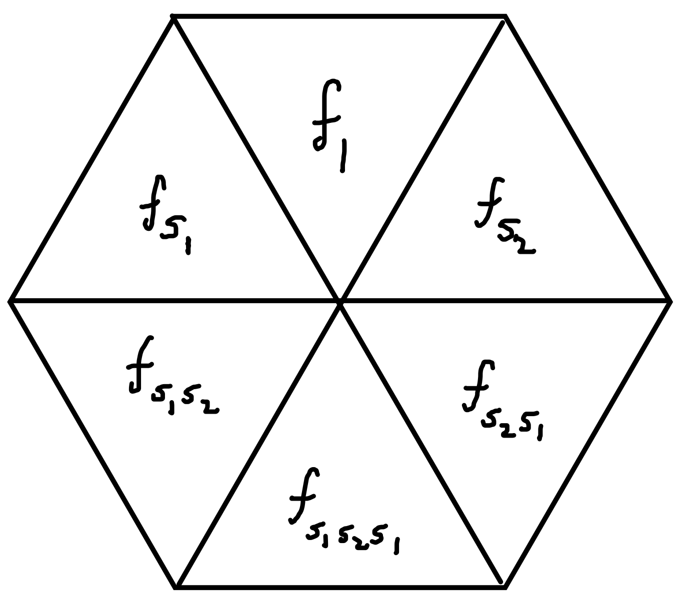


$$\in \bigoplus_{w \in W_0} S$$

$$\Omega_T(G/B) = (S \otimes S)\cdot 1 \quad \text{in} \bigoplus_{w\in W_0} S$$

$$\Omega_T(pt) = S = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu + a_{11} y_\lambda y_\mu + a_{21} y_\lambda^2 y_\mu + a_{12} y_\lambda y_\mu^2 + \cdots$$



$$\in \bigoplus_{w\in W_0} S$$

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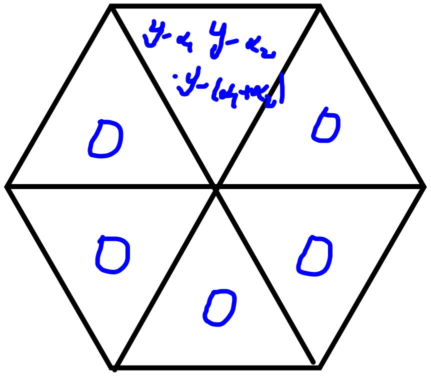
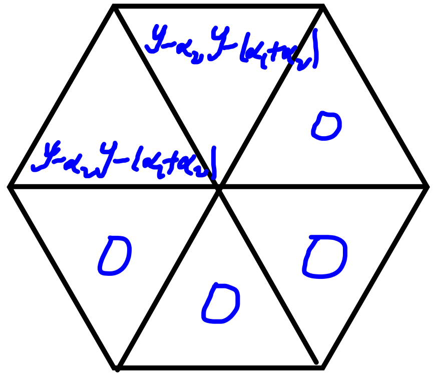
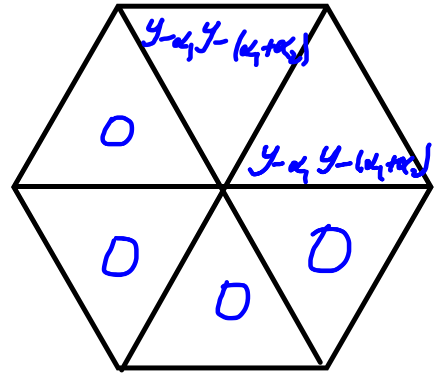


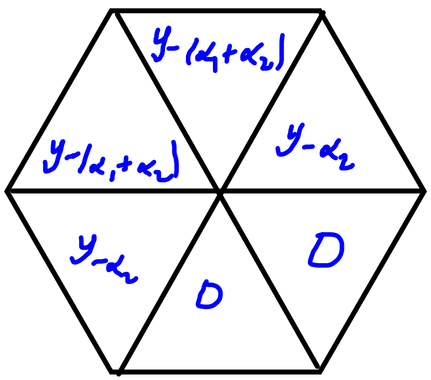
[X_{pt}]



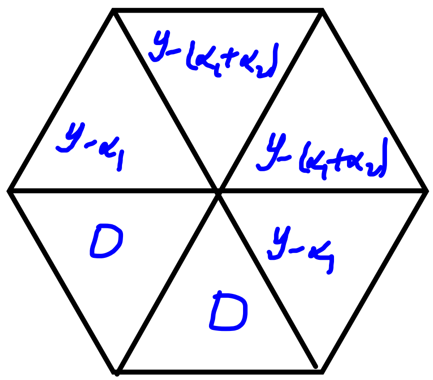
[X_{s_1}]



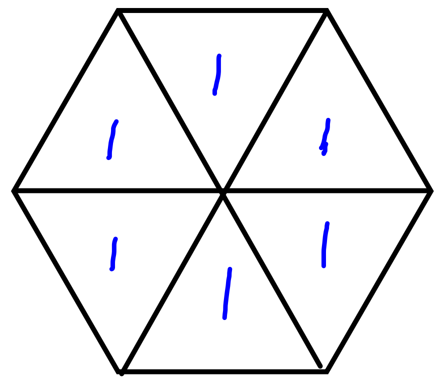
[X_{s_2}]



[$X_{s_1 s_2}$]



[$X_{s_2 s_1}$]



[$X_{s_1 s_2 s_1}$]

Schubert Classes

Conjecture Cobordism Schubert Classes

There exist unique $[X_w]$, $w \in W_0$ characterized by

(a) $\{[X_w] \mid w \in W_0\}$ is a basis of $\Omega_T(G/B)$,

(b) $[X_w]_w = \prod_{\substack{\alpha \in R^+ \\ w\alpha \notin R^+}} y_{-\alpha}$ and $[X_w]_v = 0$ unless $v \le w$.

(c) If λ is a dominant weight $\left(\lambda \in \sum_{i=1}^{n} \mathbb{Z}_{\ge 0} \omega_i\right)$

$$X_{-\lambda} [X_w] = \sum_{v \in W_0} c_{\lambda w}^{v} [X_v]$$

with $c_{\lambda w}^{v} \in \mathbb{L}_{\ge 0}[y_{-\alpha_1}, \ldots, y_{-\alpha_n}]$ where $\mathbb{L}_{\ge 0} = \mathbb{Z}_{\ge 0}[a_{11}, a_{12}, a_{21}, \ldots]$.