A probabilistic interpretation of Macdonald polynomials

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Markov chains

State space $\{w \mid w \in S_n\}$

$w =$  $\in S_5$ the symmetric group

Operator

$$M = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} s_{ij}$$

where $s_{ij} =$ , the transposition switching i and j.

Starting state: 1

The story: A deck of cards. Choose 2 cards i and j and switch them.

How long does it take to get random?

Stationary distribution

$$\pi = \frac{1}{n!} \sum_{w \in S_n} w, \quad \text{the uniform distribution}$$

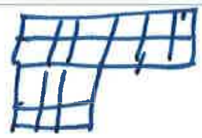
Distances to stationarity

$$4\|M^\ell 1 - \pi\|_{TV}^2 = \Big(\sum_{y \in S_n} |(M^\ell 1)(y) - \pi(y)| \Big)^2 \qquad L^1\text{-norm}$$

$$\le \quad \|M^\ell 1 - \pi\|_2^2 = \sum_{y \in S_n} \frac{\big((M^\ell 1)(y) - \pi(y)\big)^2}{\pi(y)} \qquad L^2\text{-norm}$$

Lumping

New state space $\{p_\mu \mid \mu \text{ is a partition of } n\}$

$\mu =$  $= (6,6,3,3) = 1^0 2^0 3^2 4^0 5^0 6^2$ has $n = 18$

The

$$p_\mu = \frac{z_\mu}{n!} \sum_{\tau(w)=\mu} w, \quad \text{with } \tau(w) \text{ the cycle type of } w$$

form a basis of the centre of the group algebra $\mathbb{C}S_n$.

New chain: Same as old except report $\tau(w)$.

$$M \cdot x = \sum_y M(x,y)\, y$$

$M(x,y)$ is the probability of moving from x to y.

Eigenvectors and eigenvalues

$$s_\lambda = \sum_\mu \chi^\lambda_\mu p_\mu \quad \text{and} \quad M s_\lambda = \chi^\lambda(s_{12})\, s_\lambda$$

where $\chi^\lambda_\mu = \mathrm{char}_\mu(S_n^\lambda) = \mathrm{Tr}(w, S_n^\lambda)$ are the characters of the simple S_n-modules S_n^λ.

$$\chi^\lambda(s_{12}) = \left(\begin{array}{c}\text{sum of the contents} \\ \text{of boxes of } \lambda\end{array}\right)$$

Convergence speed of M^ℓ is controlled by the second largest eigenvalue.

The Metropolis algorithm (following Hanlon...)

$\ell(\lambda) = \#$ of parts of λ

For α, $0<\alpha<1$. A step of M_α is

- if $\ell(\tau(s_{ij}w)) = \ell(\tau(w))+1$ move to $s_{ij}w$,
- if $\ell(\tau(s_{ij}w)) = \ell(\tau(w))-1$ move to $s_{ij}w$ with probability $1/\alpha$.

The new chain M_α has

stationary distribution $\pi_\alpha = \sum_\mu \alpha^{-\ell(\mu)} z_\mu p_\mu$

eigenvectors $J_\lambda^\alpha = \sum_\mu X_\mu^\lambda(\alpha) p_\mu$ Jack polynomials

eigenvalues $M_\alpha J_\lambda^\alpha = \beta_\lambda(\alpha) J_\lambda^\alpha$

where $\beta_\lambda(\alpha) = \sum_{i=1}^{n} \alpha\lambda_i + n - i$.

Unlumping to polynomials

In the world of symmetric functions

$$p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_\ell} \quad \text{for } \mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$$

where
$$p_k = x_1^k + x_2^k + \cdots + x_n^k \quad \text{for } k \in \mathbb{Z}_{>0}.$$

The operator

$$D_\alpha = \frac{\alpha}{2} \sum_{i=1}^{n} x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i}$$

acts on $\mathbb{C}[x_1, x_2, \ldots, x_n]$ with

eigenvectors J_λ^α and eigenvalues $\beta_\lambda(\alpha)$.

Now we are in the world of

Harmonic analysis: Spectra of Laplacians

Mathematical Physics: Spectra of Hamiltonians.

For $\alpha = \frac{1}{2}, 1, 2$ the J_λ^α are

the classical spherical functions (zonal polynomials)

for $\quad GL_n(\mathbb{H}) / U_n(\mathbb{H}), \quad GL_n(\mathbb{C}) / U_n(\mathbb{C}), \quad GL_n(\mathbb{R}) / O_n(\mathbb{R})$

<u>Auxiliary variables</u> = data augmentation = hit and run

Defined by Edwards and Sokal (for fast Ising and Potts)

Generalizes Swendsen-Wang

<u>The data</u>

State space $\mathcal{X}$ Auxiliary set I.

Probability distribution on $\mathcal{X}$: $\pi(x)$

Probability distribution on I: $w_x(i)$ for each $x \in \mathcal{X}$

~~Probabili~~
Markov chain on $\mathcal{X}$: $M_i(x,y)$ for each $i \in I$

such that

$$\pi(x)\, w_x(i)\, M_i(x,y) = \pi(y)\, w_y(i)\, M_i(y,x).$$

This gives a Markov chain on $\mathcal{X}$:

$$M(x,y) = \sum_i w_y(i)\, M_i(x,y)$$

Our Auxiliary variables chain

$$\mathcal{X} = \mathcal{P}_n \quad \text{and} \quad \mathcal{I} = \bigcup_{i=1}^{n} \mathcal{P}_i$$

$$\pi(\lambda) = \frac{\text{const}}{z_\lambda \prod_i \left(\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}\right)}$$

$$w_\lambda(\mu) = \frac{1}{(q^n-1)} \prod_{i=1}^{n} \binom{a_i(\lambda)}{a_i(\mu)} \frac{(q^i-1)^{a_i(\lambda)}}{(q^i-1)^{a_i(\mu)}}, \quad \text{and}$$

$$M_\rho(\lambda,\mu) = \begin{cases} \frac{1}{z_\mu(1-t^{-1})} \prod_i (1-t^{-i})^{a_i(\nu)}, & \text{if } \mu = \rho \cup \nu \\ 0, & \text{otherwise} \end{cases}$$

The story: Start with λ

- Delete some parts to get $\lambda - \gamma$ with probability $w_\lambda(\lambda-\gamma)$
- Add some parts to get μ with probability $M_{\lambda-\gamma}(\lambda,\mu)$

This gives a Markov chain

$$M_{q,t}(\lambda,\mu) \text{ on } \mathcal{P}_n = \{p_\lambda \mid \lambda \text{ is a partition of } n\}$$

Macdonald polynomials

Theorem The eigenvectors of $M_{q,t}$ are

$$P_\lambda(q,t) = \sum_\mu \chi_\mu^\lambda(q,t)\, g_\mu$$

the Macdonald polynomials, and

$$M_{q,t} P_\lambda(q,t) = \beta_\lambda(q,t)\, P_\lambda(q,t)$$

where $$\beta_\lambda(q,t) = \sum_{i=1}^{\ell(\lambda)} q^{\lambda_i} t^{n-i}$$

Remarks

- $P_\lambda(0,0) = s_\lambda$ = Schur functions
  = characters of compact Lie groups
- $P_\lambda(0,t)$ = Hall-Littlewood polynomials
  = spherical functions for $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$
- $\lim_{t\to 1} P_\lambda(t^\alpha, t) = J_\lambda^\alpha$, the Jack polynomials
- For type $(C_n^\vee, C_n)$, $P_\lambda(q,t)$ are the Koornwinder polynomials
- For type $(C_1^\vee, C_1)$, $P_\lambda(q,t)$ are the Askey-Wilson polynomials