Pure math seminar
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What is a Schubert polynomial?

A Schubert polynomial is the class $[X_w]$
of a Schubert variety X_w
in $H_T(G/B)$, where

G/B is the flag variety

H_T is T-equivariant generalised cohomology.

G = complex reductive algebraic group $\qquad GL_n(\mathbb{C})$

$\cup\!|\qquad\qquad\qquad\qquad \cup\!|$

B = Borel subgroup $\qquad \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$

$\cup\!|\qquad\qquad\qquad\qquad \cup\!|$

T = maximal torus $\qquad \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$

Then

$$G = \bigsqcup_{w \in W_0} BwB \quad \text{where} \quad W_0 = N(T)/T.$$

If $G = GL_n(\mathbb{C})$ then $W_0 = S_n = \left\{\begin{matrix} n\times n \text{ permutation} \\ \text{matrices} \end{matrix}\right\}$.

The Schubert varieties are

$$X_w = \overline{BwB} \quad \text{in } G/B \quad \text{for } w \in W_0$$

where gB and hB are close if g and h are close.

The ring $H_T(pt)$:

W_0 acts on the weight lattice

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}(T, \mathbb{C}^\times) = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n.$$

If $G = GL_n(\mathbb{C})$ then $\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ with $W_0 = S_n$ acting by permuting $\varepsilon_1, \ldots, \varepsilon_n$.

Let

$$y_i = y_{\varepsilon_i} \quad \text{for } i = 1, 2, \ldots, n.$$

A formal group law is a_{ij}, $i,j \in \mathbb{Z}_{\geq 1}$, such that if

$$y_{\lambda+\mu} = y_\lambda \oplus y_\mu = y_\lambda + y_\mu + a_{11} y_\lambda y_\mu + a_{12} y_\lambda y_\mu^2 + \cdots$$

then

$$y_\lambda \oplus (y_\mu \oplus y_\nu) = (y_\lambda \oplus y_\mu) \oplus y_\nu,$$

$$y_\lambda \oplus y_\mu = y_\mu \oplus y_\lambda, \qquad y_\lambda \oplus y_{-\lambda} = y_0 = 0.$$

Let

$$H_T(pt) = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]] = \mathbb{C}[[y_1, \ldots, y_n]].$$

Examples

(1) Ordinary cohomology: $a_{ij} = 0$ for $i,j \in \mathbb{Z}_{\geq 1}$.

$y_{\lambda+\mu} = y_\lambda + y_\mu$ and $H_T(pt) = \mathbb{C}[y_1, \ldots, y_n]$

(2) K-theory: $a_{ij} = 0$ for $i,j \in \mathbb{Z}_{>1}$

$y_{\lambda+\mu} = y_\lambda + y_\mu - y_\lambda y_\mu$ and $H_T(pt) = K_T(pt) = Rep(T)$.

(3) Elliptic cohomology: a_{ij} are determined by the group law on an elliptic curve

$$E_\tau = \frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}} \quad \text{with } \tau \in \mathbb{C},\ \tau = x + iy \text{ and } y \in \mathbb{R}_{>0}$$

(4) Cobordism: The universal formal group law: The Lazard ring is the ring generated by a_{ij}, $i,j \in \mathbb{Z}_{\geq 1}$, subject to the relations given by (FG1) and (FG2). Then

$$H_T(pt) = \Omega_T(pt).$$

Cohomology of the flag variety $H_T(G/B)$

Borel model: The fibration sequence

$$G/B \to BT \to BG \qquad \text{and} \qquad \begin{aligned} H_T(pt) &= H(BT) \\ H_G(pt) &= H_T(pt)^{W_0}. \end{aligned}$$

give

$$H_T(G/B) = H_T(pt) \otimes_{H_G(pt)} H_T(pt)$$

$$= \frac{\mathbb{C}[[x_1, \ldots, x_n, y_1, \ldots, y_n]]}{\langle f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \mid f \in \mathbb{C}[[y_1, \ldots, y_n]]^{W_0} \rangle}$$

Moment graph model The T-fixed points in G/B

$$\begin{aligned} \iota_v : pt &\longrightarrow G/B \\ * &\longmapsto vB \end{aligned} \qquad \text{give} \qquad \iota_v^* : H_T(G/B) \longrightarrow H_T(pt)$$

and

$$H_T(G/B) \xrightarrow{\oplus_v \iota_v^*} \bigoplus_{v \in W_0} H_T(pt) \quad \text{is injective}$$

and

$$H_T(G/B) = \left\{ (f_v)_{v \in W_0} \;\middle|\; \begin{array}{l} f_v \in \mathbb{C}[[y_1, \ldots, y_n]] \text{ and if } s_\alpha \in R^+ \\ f_v - f_{vs_\alpha} \in y_\alpha \mathbb{C}[[y_1, \ldots, y_n]] \end{array} \right\}$$

where $R^+ = \{ s_\alpha \in W_0 \mid s_\alpha \text{ is a reflection} \}$

$$(\mathfrak{h}^*_{\mathbb{R}})^{s_\alpha} = \mathbb{R}\alpha.$$

The Schubert polynomials $[X_w]$.

Case 1: X_w is smooth $\left(\begin{array}{l}\dim(T_p(X_w)) \text{ is the} \\ \text{same for all } p \in X_w\end{array}\right)$

$j_w : X_w \hookrightarrow G/B$ and $(j_w)_! : H_T(X_w) \longrightarrow H_T(G/B)$

Then

$$[X_w] = (j_w)_!(1) \quad \text{so that} \quad [X_w] = ([X_w]_v)_{v \in W_0}$$

with

$$[X_w]_v = \begin{cases} \prod\limits_{\substack{\alpha \in R^+ \\ w\alpha \in R^+}} y_{-v\alpha}, & \text{if } BvB \subseteq \overline{BwB} \\ 0, & \text{if } BvB \not\subseteq \overline{BwB} \end{cases}$$

Case 2: X_w is not smooth

A resolution of singularities is

$$\begin{tikzcd} & Z \arrow[dl, "\gamma"'] \arrow[dr, "k"] & \\ X_w \arrow[rr, hook, "j_w"'] & & G/B \end{tikzcd}$$

with (a) Z is smooth

(b) $\gamma : Z \to X_w$ is birational.

Let

$$[Z] = k_!(1) \quad \text{where} \quad k_! : H_T(Z) \longrightarrow H_T(G/B).$$

For ordinary cohomology and K-theory

$[Z]$ does not depend on the choice of Z.

Bott-Samelson resolutions

$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ are the simple roots and

$s_i = $ (strands $1\ 2 \cdots i\ i{+}1 \cdots n$, with strands i and $i+1$ crossed) are the simple reflections.

Let $w = s_{i_1} \cdots s_{i_\ell}$ be a minimal length expression for w.

$$\overline{Bs_{i_1}B} \times_B \overline{Bs_{i_2}B} \times_B \cdots \times_B \overline{Bs_{i_\ell}B} = Z_{i_1 \cdots i_\ell}$$

$$(p_1B, p_2B, \ldots, p_\ell B) \longmapsto p_1p_2\cdots p_\ell B.$$

$$\begin{tikzcd} & Z_{i_1\cdots i_\ell} \arrow[dl, "\gamma_{i_1\cdots i_\ell}"'] \arrow[dr, "\kappa_{i_1\cdots i_\ell}"] & \\ X_w \arrow[rr, hook, "j_w"'] & & G/B \end{tikzcd}$$

Then

$$[Z_{i_1\cdots i_\ell}] = (\kappa_{i_1\cdots i_\ell})_*(1) = A_{i_1}\cdots A_{i_\ell}[X_1]$$

where

$$[X_1]_v = \begin{cases} \prod_{\alpha\in R^+} y_{-\alpha}, & \text{if } v=1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$A_i = (1 + t_{s_i})\frac{1}{x_{-\alpha_i}}$$

with

$$(t_{s_i}f)_v = f_{s_iv} \quad \text{and} \quad \left(\frac{1}{x_{-\alpha_i}}\right)_v = \frac{1}{y_{-v\alpha_i}}.$$