Talk at Taipei Conference on Representation Theory IV
20–23 December 2013 (1)

Fock space and representations of quantum groups

Hayashi
Misra-Miwa
} semi infinite q-wedges
$u_{i_1}\wedge u_{i_2}\wedge\cdots$ with $i_1>i_2>\cdots$, $i_j\in\mathbb{Z}$
all but a finite part equal to $u_0\wedge u_{-1}\wedge u_{-2}\wedge\cdots$
$U_q\hat{sl}_\ell$-action

Kashiwara-Miwa-Stern
Leclerc-Thibon
Leclerc
} Hecke algebra/Schur-Weyl construction
q-skew symmetrize $\mathbb{C}^\ell[z^{\pm1}]^{\otimes k}$ and $k\to\infty$

Ariki, LLT
Kleshchev
} representations of Hecke algebras and affine Hecke algebras at roots of 1
addable/removable box combinatorics

Lusztig
Andersen; Kato
Kaneda
Soergel
} "Jantzen's generic decomposition"
inverse affine KL polys
Kashiwara-Tanisaki
Kazhdan-Lusztig.

Ram-Tingley } Misra-Miwa action of $U_q\hat{sl}_\ell$ measures Jantzen depth on translation by $\mathbb{C}^n$ for $U_\epsilon gl_n$-modules.

<u>Goal</u>: Unify these and give a general type combinatorial model.

The space $\mathcal{F}_{-\ell}$

Weyl character formula for fin. dim'l simple $\mathfrak{g}_0$.

$\mathring{\mathfrak{h}}^*_{\mathbb{Z}}$ = weight lattice $\qquad W_0$ = Weyl group

ρ such that $\langle \rho, \alpha_i^\vee \rangle = 1$ for simple coroots $\alpha_1^\vee, \ldots, \alpha_n^\vee$

$s_1, \ldots, s_n$ simple reflections in W_0

$(\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+$ = dominant integral weights

$$w \circ \lambda = w(\lambda+\rho) - \rho \qquad \text{(dot action)}$$

The Weyl character is

$$s_\lambda = \frac{\sum_{w \in W_0} (-1)^{\ell(w)} X^{w \circ \lambda}}{\sum_{w \in W_0} (-1)^{\ell(w)} X^{w \circ \rho}} \qquad \text{for } \lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+$$

Let $\ell \in \mathbb{Z}_{>0}$ and $\mathbb{A} = \mathbb{Z}[t^{\pm\frac{1}{2}}]$

$$\mathcal{F}_{-\ell} = \mathbb{A}\text{-span}\{ |\lambda\rangle \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}} \} \quad \text{with}$$

$$|\lambda\rangle = \begin{cases} -|s_i \circ \lambda\rangle, & \text{if } \langle \lambda+\rho, \alpha_i^\vee \rangle \in \ell\mathbb{Z}_{\le 0} \\ -t^{\frac{1}{2}} |s_i \circ \lambda\rangle, & \text{if } -\ell < \langle \lambda+\rho, \alpha_i^\vee \rangle < 0 \\ -|\lambda^{(1)}\rangle - t^{\frac{1}{2}} |s_i \circ \lambda^{(1)}\rangle - t^{\frac{1}{2}} |s_i \circ \lambda\rangle, & \text{otherwise} \end{cases}$$

where $\lambda^{(1)} = \lambda + j\alpha_i$ if $\langle \lambda+\rho, \alpha_i^\vee \rangle = -k\ell - j$ with $0 < j < \ell$.

KL basis of $\mathcal{F}_{-\ell}$ and $U_\epsilon\mathfrak{g}$-mod

$\mathcal{F}_{-\ell}$ has bar involution given by $\overline{t^{\frac12}} = t^{-\frac12}$ and

$$\overline{|\lambda\rangle} = (t^{\frac12})^{\ell(w_\lambda)} (-t^{-\frac12})^{\ell(w_0)} |w_0 \circ \lambda\rangle$$

where w_0 = longest element of W_0 and

w_λ = longest element of $\mathrm{Stab}_{W_0}(\lambda)$.

Then $\mathcal{F}_{-\ell}$ has standard basis $\{|\lambda\rangle \mid \lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+\}$

and KL-basis $\{C_\lambda \mid \lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+\}$ given by

$$\overline{C_\lambda} = C_\lambda \quad \text{and} \quad C_\lambda = |\lambda\rangle + \sum_\mu d_{\mu\lambda} |\mu\rangle$$

with $d_{\mu\lambda} \in t^{\frac12}\mathbb{Z}[t^{\frac12}]$. Then

$$\begin{array}{ccll}
\mathcal{F}_{-\ell} & \longrightarrow & \text{"graded" Grothendieck group fin. dim'l } U_\epsilon\mathfrak{g}\text{-mod}, \ \epsilon^{\ell}=1 & \\
|\lambda\rangle & \longmapsto & \Delta_\epsilon(\lambda) & \text{Weyl module h.w. } \lambda \\
C_\lambda & \longmapsto & L_\epsilon(\lambda) & \text{simple module hw } \lambda.
\end{array}$$

meaning, if $\langle\ \rangle$ is a Shapovalov form on $\Delta(\lambda)$ and

$$\Delta_\epsilon(\lambda) = \Delta_\epsilon(\lambda)^{(0)} \supseteq \Delta_\epsilon(\lambda)^{(1)} \supseteq \cdots$$

is then corresponding Jantzen filtration and

$$Q_{\mu\lambda}(t^{\frac12}) = \sum_j (t^{\frac12})^j \left[\Delta_\epsilon(\lambda)^{(j)} / \Delta_\epsilon(\lambda)^{(j+1)} : L(\mu) \right]$$

then $|\lambda\rangle = \sum_\mu Q_{\mu\lambda}(t^{\frac12}) C_\mu$.

Affine polynomial representation of H

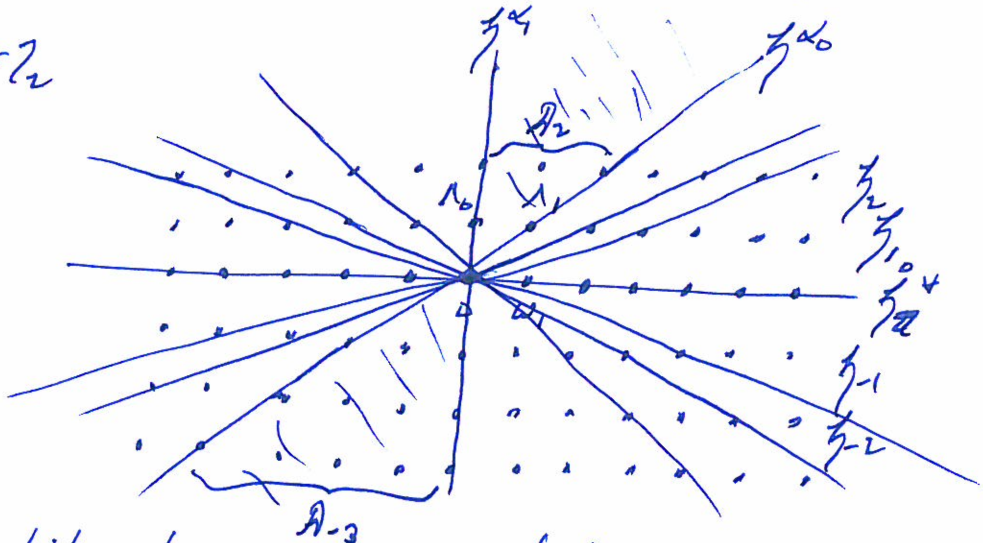
$\mathfrak{g} = \mathbb{C}K \oplus \mathring{\mathfrak{g}}[z, z^{-1}] \oplus \mathbb{C}d$ affine Lie algebra

W = affine Weyl group acts on

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\delta \oplus \mathring{\mathfrak{h}}_{\mathbb{Z}}^* \oplus \mathbb{Z}\Lambda_0 = \bigsqcup_{\ell \in \mathbb{Z}} \mathfrak{h}_\ell^*.$$

where $\mathfrak{h}_\ell^* = \{a\delta + \bar{\lambda} + \ell\Lambda_0 \mid a \in \mathbb{Z}, \bar{\lambda} \in \mathring{\mathfrak{h}}_{\mathbb{Z}}^*\}$

If $\mathring{\mathfrak{g}} = \mathfrak{sl}_2$



The W-orbits have representatives

$$a\delta + v \quad \text{with } v \in A_\ell$$

H = affine Hecke algebra = $\mathbb{A}$-span$\{T_w \mid w \in W\}$

acts on

$$P = \bigoplus_{\ell \in \mathbb{Z}} P_\ell \quad \text{where} \quad P_\ell = \bigoplus_{\substack{a \in \mathbb{Z} \\ v \in A_\ell}} H \mathbb{1}_v$$

where $T_w \mathbb{1}_v = (t^{\frac{1}{2}})^{\ell(w)} \mathbb{1}_v$ for $w \in \mathrm{Stab}_W(v)$.

Lusztig conjecture

Define

$$\widetilde{\mathcal{F}}_{\ell} = \varepsilon_0 P_{-\ell} = \bigoplus_{\substack{v \in A_{-\ell} \\ a \in \mathbb{Z}}} \varepsilon_0 H \mathbb{1}_v$$

with $\varepsilon_0 T_w = (-t^{-\frac{1}{2}})^{\ell(w)} \varepsilon_0$ for $w \in W_0$ and use

bar involution $\overline{\varepsilon_0 h \mathbb{1}_v} = \varepsilon_0 \bar{h} \mathbb{1}_v$ for $h \in H$.

Let

$$[\lambda] = [w \circ v] = \varepsilon_0 X^w \mathbb{1}_v \qquad \text{for } \lambda = w \circ v \in \mathfrak{h}^*_{\ell}$$

where $X^w = X^\gamma T_{v^{-1}}^{-1}$ if $w = t_\gamma v$ (Bernstein generators).

Then $\Phi(\overline{[\lambda]}) = \overline{\Phi([\lambda])}$

$$\begin{array}{ccccccc}
\mathcal{F}_{-\ell} & \xrightarrow{\Phi} & \widetilde{\mathcal{F}}_{-\ell} & \xrightarrow{\sim} & \begin{array}{c}\text{"graded" Groth. gp.} \\ \mathfrak{g}_0\text{-parabolic} \\ \text{category } \mathcal{O} \text{ for } \mathfrak{g}.\end{array} & \longrightarrow & \begin{array}{c}\text{"graded" Groth. gp} \\ \text{fin. dim. } U_\varepsilon\mathfrak{g}\text{-mod} \\ \varepsilon^\ell = 1\end{array} \\
|\lambda\rangle & \longmapsto & [\lambda] & \longmapsto & \Delta^{\mathfrak{g}}_{\mathfrak{g}_0}(\lambda) & \longmapsto & \Delta_\varepsilon(\lambda) \\
C_\lambda & \longmapsto & C_\lambda & \longmapsto & L_{\mathfrak{g}}(\lambda) & \longmapsto & L_\varepsilon(\lambda)
\end{array}$$

where $\Delta^{\mathfrak{g}}_{\mathfrak{g}_0}(\lambda) = U\mathfrak{g} \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{n}^+)} L_{\mathfrak{g}_0}(\lambda)$.

Then

$$C_\lambda = C_{w \circ v} = \sum_{v \in W} (-1)^{\ell(w) + \ell(v)} P_{vw}(t^{\frac{1}{2}}) \, |v \circ v\rangle$$

where $P_{vw}(t^{\frac{1}{2}})$ are the KL polynomials for H.

<u>Lusztig-Steinberg tensor product</u> $\mathbb{A} = \mathbb{Z}[t^{\pm\frac{1}{2}}]$

Define a right action of

$$\mathbb{A}[X] = \mathbb{A}\text{-span}\{X^\mu \mid \mu \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}\} \quad \text{on} \quad \mathcal{F}_{-\ell} \quad \text{by}$$

$$|\lambda\rangle X^\mu = |\lambda + \ell\mu\rangle.$$

Let $\lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+$ and

$$\lambda = \lambda^{(0)} + \ell\lambda^{(1)} \quad \text{with} \quad \lambda^{(0)} \in \Pi_\ell \quad \text{and} \quad \lambda^{(1)} \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+$$

where $\Pi_\ell = \{a_1\omega_1 + \cdots + a_n\omega_n \mid a_i \in \{0,1,\ldots,\ell-1\}\}$. Then

$$(*) \qquad C_\lambda = C_{\lambda^{(0)}} s_{\lambda^{(1)}}.$$

is a decategorification of $L_\epsilon(\lambda) = L_\epsilon(\lambda^{(0)}) \otimes F^*(\Delta(\lambda^{(1)}))$

where $F: U_\epsilon\mathfrak{g} \to U\mathfrak{g}$ is the "Frobenius" map.

If $\ell = 1$:

$$\begin{array}{ccccc}
\mathbb{A}[X]^{W_0} = Z(H) & \xrightarrow[\sim]{\text{Satake}} & \mathbb{1}_0 H \mathbb{1}_0 & \xrightarrow[\sim]{\text{Lusztig}} & \varepsilon_0 H \mathbb{1}_0 = \mathcal{F}_{-1} \\
h & \longmapsto & h\mathbb{1}_0 & \longmapsto & |0\rangle h \mathbb{1}_0 \\
s_\lambda & \longleftarrow\!\shortmid & C'_\lambda & \longleftarrow\!\shortmid & |\lambda\rangle = \varepsilon_0 X^{\lambda+\rho}\mathbb{1}_0
\end{array}$$

(see Lusztig "q-singularities" and Nelsen-Ram)

<u>Point:</u> View (*) as the ℓ-analogue of Satake-Lusztig.