Abelian varieties and elliptic curves

An *abelian variety* is a complex torus $\mathbb{C}^g/\Lambda(\cong\mathbb{Z}^{2g})$ which imbeds in projective space.

An *elliptic curve* is an abelian variety of dimension 1.

$g=1$	general g
$\left\{\begin{array}{l}\text{complex tori}\\ \text{of dim } 1\end{array}\right\}$	$\left\{\begin{array}{l}\text{complex tori}\\ \text{of dim } g\end{array}\right\}$
$\parallel$	$\cup\!\!\!\!\not\;$
$\left\{\begin{array}{l}\text{elliptic}\\ \text{curves}\end{array}\right\}$	$\left\{\begin{array}{l}\text{abelian}\\ \text{varieties}\end{array}\right\}$

The case $g=1$ (see, for example, [AEC II Ch. 1])

~~Let~~ The *upper half plane* is

$$\mathfrak{S}_1 = \{\tau\in\mathbb{C} \mid \operatorname{Im}(\tau)\in\mathbb{R}_{>0}\}$$

Then

$$\mathfrak{S}_1 \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$$

The *moduli space of elliptic curves* is

$$X(1) = \Gamma(1)\backslash\mathfrak{S}_1 = \Gamma(1)\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R}) = SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$$

where $\Gamma(1) = \dfrac{SL_2(\mathbb{Z})}{\{\pm1\}} = PSL_2(\mathbb{Z})$.

<u>The general g case</u>: [Igusa p24 and Ch. I Theorem 5].

The <u>Siegel upper half space</u> is

$$\mathfrak{S}_g = \{\tau \in M_g(\mathbb{C}) \mid \tau = \tau^t,\ \mathrm{Im}(\tau) \text{ is positive definite}\}.$$

Then

(a) $\mathfrak{S}_g \simeq Sp_{2g}(\mathbb{R}) \big/ Sp_{2g}(\mathbb{R}) \cap O_{2g}(\mathbb{R})$

(b) $K = Sp_{2g}(\mathbb{R}) \cap O_{2g}(\mathbb{R})$ is a maximal compact subgroup of $Sp_{2g}(\mathbb{R})$.

(c) All maximal compact subgroups of $Sp_{2g}(\mathbb{R})$ are conjugate.

Let $d_1, \ldots, d_g \in \mathbb{Z}_{>0}$ with $d_1 | d_2, d_2 | d_3, \ldots, d_{g-1} | d_g$. Let

$$\Delta = \left(\begin{array}{c|c} 0 & \begin{matrix} d_1 & & \\ & \ddots & \\ & & d_g \end{matrix} \\ \hline \begin{matrix} -d_1 & & \\ & \ddots & \\ & & -d_g \end{matrix} & 0 \end{array}\right) \quad \text{and}$$

$$Sp(\Delta, \mathbb{Z}) = \{M \in GL_{2g}(\mathbb{Z}) \mid M \Delta M^t = \Delta\}$$

acting on $\mathfrak{S}_g$ by

$$M \cdot \tau = (A\tau + B\Delta)(C\tau + D\Delta)^{-1}, \qquad \text{if } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then

$$\left\{\begin{matrix}\text{isom. classes of}\\ \text{abelian varieties}\\ \text{with } \Delta \text{ polarization}\end{matrix}\right\} \xrightarrow{\sim} Sp(\Delta,\mathbb{Z})\backslash \mathfrak{S}_g \xrightarrow{\sim} Sp(\Delta,\mathbb{Z})\backslash Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{R})\cap O_{2g}(\mathbb{R})$$

$$\left(\mathbb{C}^g/\Lambda\binom{\tau}{1}, \mathcal{L}\right) \longleftarrow\!\shortmid \ \tau$$

Elliptic functions talk
06.03.2014

Modular forms

Let $k \in \mathbb{Z}_{\geq 2}$. The Eisenstein series of weight $2k$ is

$$G_{2k}(\Lambda) = \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \frac{1}{\omega^{2k}}$$

Let $G_k(\Lambda)=0$, for k odd. Let $\Gamma(1) = PSL_2(\mathbb{Z}) = \dfrac{SL_2(\mathbb{Z})}{\{\pm 1\}}$

Write

$$M_{2k} = \{\text{modular forms of weight } 2k \text{ for } \Gamma(1)\}$$

$$M_{2k}^{\circ} = \{\text{cusp forms of weight } 2k \text{ for } \Gamma(1)\}$$

Then

$$M_{2k} \text{ has basis } \{G_4^a G_6^b \mid a,b \in \mathbb{Z}_{\geq 0},\ 2a+3b=k\}$$

$$M_{2k} = M_{2k}^{\circ} + \mathbb{C}G_{2k} \quad \text{and} \quad \begin{array}{ccc} M_{2k}^{\circ} & \hookleftarrow & M_{2k-12} \\ f\Delta & \longmapsfrom & f \end{array}$$

where $\Delta = g_2^3 - 27g_3$ with $g_2 = 60G_4$ and $g_3 = 140G_6$.

In particular,

$$\dim M_k = 0, \text{ if } k \in \mathbb{Z}_{<0} \text{ or } k \text{ odd or } k=2,$$

$$\dim M_k = 1, \text{ if } k \in \{0, 4, 6, 8, 10, 14\}$$

and

$$\dim(M_{2k+12}) = \dim(M_{2k}) + 1, \text{ for } k \in \mathbb{Z}_{\geq 0}.$$

Let $k \in \mathbb{Z}$. A <u>weakly</u> <u>modular</u> <u>function</u> <u>of weight</u> k <u>for</u> $\Gamma(1)$ is $f : G_1 \to \mathbb{C}$ such that

(a) If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ and $\tau \in G_1$ then

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

(b) f is meromorphic on G_1.

<u>Note</u>: (a) Since $f(\tau+1) = f(\tau)$ then f has a Fourier expansion

$$f = \sum_{n \in \mathbb{Z}} a_n q^n, \text{ where } q = e^{2\pi i \tau} \quad \left(\text{so } f \in \ell^2(\mathbb{Z})?\right)$$

(b) If k is odd then the only weakly modular function is $f = 0$.

A <u>modular function</u> is a weakly modular function f such that

$$f = \sum_{n=-n_0}^{\infty} a_n q^n \quad (\text{i.e. } f \in \mathbb{C}((q))).$$

A <u>modular form</u> is a weakly modular function f such that

$$f = \sum_{n \in \mathbb{Z}_{\geq 0}} a_n q^n \in \mathbb{C}[[q]].$$

A <u>cusp form</u> is a weakly modular function f such that

$$f = \sum_{n \in \mathbb{Z}_{>0}} a_n q^n \in q\,\mathbb{C}[[q]].$$

Let
$$M_{2k} = \{\text{modular forms of weight } 2k \text{ for } \Gamma(1)\}$$
$$M^\circ_{2k} = \{\text{cusp forms of weight } 2k \text{ for } \Gamma(1)\}.$$

<u>Theorem</u>

(a) [AEC II Ch. I Ex 1.10]

$$\begin{array}{ccc} \mathbb{C}[X,Y] & \xrightarrow{\sim} & \bigoplus_{k \in \mathbb{Z}_{\geq 0}} M_k \\ X & \longmapsto & G_4 \\ Y & \longmapsto & G_6, \end{array} \quad \text{with } \begin{array}{l} \deg(X) = 2 \\ \deg(Y) = 3 \end{array}$$

so

M_{2k} has basis $\{G_4^a G_6^b \mid a, b \in \mathbb{Z}_{\geq 0},\ 2a + 3b = k\}$.

(b)
$$M_{2k} = M^\circ_{2k} + \mathbb{C}G_{2k}$$

(since $M^\circ_{2k} = \ker\begin{pmatrix} M_{2k} \to \mathbb{C} \\ f \mapsto a_0 \end{pmatrix}$ so $\dim\left(M_{2k}/M^\circ_{2k}\right) \leq 1$)

(c)
$$\begin{array}{ccc} M_{2k-12} & \xrightarrow{\sim} & M^\circ_{2k} \\ f & \longmapsto & f\Delta, \end{array}$$
where $\Delta = g_2^3 - 27g_3$
with $g_2 = 60G_4$ and $g_3 = 140G_6$

(d)
$$\dim(M_{2k}) = \begin{cases} 0, & \text{if } k \in \mathbb{Z}_{<0} \\ \left[\frac{k}{6}\right], & \text{if } k \in \mathbb{Z}_{\geq 0} \text{ and } k \equiv 1 \bmod 6. \\ \left[\frac{k}{6}\right] + 1, & \text{if } k \in \mathbb{Z}_{\geq 0} \text{ and } k \not\equiv 1 \bmod 6. \end{cases}$$

(CAN WE SAY THIS BETTER?)

$2k$	0	2	4	6	8	10	12	14	16	18	20	22	24	26
$\dim(M_{2k})$	1	0	1	1	1	1	2	1	2	2	2	2	3	2
$\dim(M^0_{2k})$	0	0	0	0	0	0	1	0	1	1	1	1	2	1

~~So $\cancel{\dim(M_0)=1,\ \dim(M_2)=0,\ \dim(M_4)=1}$~~

~~$\dim M_k = 0$ if $k=2$ or k odd or $k \in \mathbb{Z}_{<0}$.~~

~~$\dim(M_k) = \begin{cases} 1, & \text{if } k \in \{0,4,6,8,10\} \\ \cancel{0,} & \cancel{\text{if } k \in \{\ldots\}} \end{cases}$~~

~~and~~

~~$\dim (M_{2k+12}) = \dim(M_{2k})+1.$~~

So

$\dim(M_k) = 0$, if $k \in \mathbb{Z}_{<0}$ or k odd or $k=2$,

$\dim(M_k) = 1$, if $k \in \{0,4,6,8,10,14\}$

and

$\dim(M_{2k+12}) = \dim(M_{2k})+1$, for $k \in \mathbb{Z}_{\geq 0}$.

Expansion of G_{2k}

Let B_k be the Bernoulli numbers given by

$$\frac{x}{e^x-1} = \sum_{k\in\mathbb{Z}_{\geq 0}} B_k \frac{x^k}{k!}$$

Let $\zeta(s)$ be the Riemann zeta function given by

$$\zeta(s) = \sum_{n\in\mathbb{Z}_{>0}} \frac{1}{n^s}.$$

Let $\sigma_k(n) = \sum_{d|n} d^k$, for $n\in\mathbb{Z}_{>0}$ and $k\in\mathbb{Z}_{\geq 0}$.

Then

$$\begin{aligned} G_{2k}(\tau) &= \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \frac{1}{\omega^{2k}} \\ &= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n\in\mathbb{Z}_{>0}} \sigma_{2k-1}(n) q^n \\ &= 2\zeta(2k) E_{2k}(\tau) \end{aligned}$$

and

$$\zeta(2k) = -\frac{(2\pi i)^{2k}}{2(2k)!} B_{2k}$$

The discriminant Δ and j

$$\Delta = g_2^3 - 27 g_3, \quad \text{where } g_2 = 60 G_4 \text{ and } g_3 = 140 G_6$$

$$= (2\pi)^{12} q \prod_{n \in \mathbb{Z}_{\geq 1}} (1-q^n)^{24} = (2\pi)^{12} \eta(\tau)^{24} \quad \left(\begin{array}{l}\text{Dedekind} \\ \text{eta-function}\end{array}\right)$$

$$= (2\pi)^{12} \sum_{n \in \mathbb{Z}_{>0}} \tau(n) q^n \qquad (\text{Ramanujan } \tau\text{-function}).$$

$$= (2\pi)^{12} (q - 24q^2 + 252 q^3 - 1472 q^4 + 4830 q^5 + \cdots)$$

The modular j-invariant

$$j(\tau) = 1728 \frac{g_2(\tau)^3}{\Delta(\tau)}$$

$$= q^{-1} + \sum_{n \in \mathbb{Z}_{\geq 0}} c(n) q^n$$

$$= q^{-1} + 744 + 196884 q + 21493760 q^2 + \cdots$$

<u>Elliptic functions</u> [WW Ch 20] and [AEC I, Ch VI]

Let Λ be a rank 2 lattice in $\mathbb{C}$,

$$\Lambda = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\}.$$

An <u>elliptic function relative to Λ</u> is a meromorphic function $f: \mathbb{C} \to \mathbb{C}$ such that

if $\omega \in \Lambda$ and $z \in \mathbb{C}$ then $f(z+\omega) = f(z)$.

Let

$$\mathbb{C}(\Lambda) = \{\text{elliptic function relative to } \Lambda\}.$$

The <u>Weierstrass</u> <u>$\wp$-function</u> is

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

Let $k \in \mathbb{Z}_{\geq 2}$. The <u>Eisenstein</u> <u>series</u> <u>of weight $2k$</u> is

$$G_{2k}(\Lambda) = \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \frac{1}{\omega^{2k}}$$

Let $G_k(\Lambda) = 0$, for k odd, and let

$$g_2 = 60\,G_4 \quad \text{and} \quad g_3 = 140\,G_6.$$

<u>Theorem</u>

(a) $\mathbb{C}(\Lambda) = \mathbb{C}(\wp(z), \wp'(z))$

(b) $$\wp(z) = z^{-2} + \sum_{k \in \mathbb{Z}_{>0}} (2k+1) G_{2k+2} z^{2k}$$

(c) $$(\wp'(z))^2 = 4\wp(z) - g_2 \wp(z) - g_3 .$$

(d) $$\begin{aligned} \mathbb{C}/\Lambda &\xrightarrow{\sim} E \subseteq \mathbb{P}^2(\mathbb{C}) \\ z &\longmapsto [\wp(z), \wp'(z), 1] \end{aligned}$$

where $E = \{[x, y, 1] \in \mathbb{P}^2(\mathbb{C}) \mid y^2 = 4x^3 - g_2 x - g_3\}$.
(FIX THIS SO IT REALLY LIES IN $\mathbb{P}^2(\mathbb{C})$).
The roots of the polynomial are given by

$$4x^3 - g_2 x - g_3 = 4(x - \wp(\omega_1))(x - \wp(\omega_2))(x - \wp(-\omega_1 - \omega_2))$$

(see [AEC II Ch. 1 Proof of Theorem 8.1] and [WW 20.32])
and the discriminant of $4x^3 - g_2 x - g_3$ is

$$\Delta(\Lambda) = g_2^3 - 27 g_3^2 = \begin{vmatrix} 1 & 1 & 1 \\ \wp(\omega_1) & \wp(\omega_2) & \wp(\omega_3) \\ \wp(\omega_1)^2 & \wp(\omega_2)^2 & \wp(\omega_3)^2 \end{vmatrix}^2$$

(what does discriminant mean??).

The Weierstrass ζ-function

(see [WW §20.4] and [AEC II, §5]).

Define $\zeta(z;\Lambda)$ by

$$\frac{d}{dz}\zeta(z;\Lambda) = -\wp(z;\Lambda).$$

Then

$$\zeta(z;\Lambda) = \frac{1}{z} + \sum_{\substack{\omega\in\Lambda\\ \omega\neq 0}}\left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right)$$

$$= \frac{1}{z} - \sum_{k\in\mathbb{Z}_{>0}} G_{2k+2}(\Lambda)\, z^{2k+1}$$

$$= \frac{1}{z} - \sum_{k\in\mathbb{Z}_{\geq 2}} G_{k+1}(\Lambda)\, z^{k}, \quad \text{since } G_k(\Lambda)=0 \text{ for odd } k.$$

The Weierstrass σ-function (see [AEC II, Prop. 5.4])

Define $\sigma(z;\Lambda)$ by

$$\sigma(z;\Lambda) = z\prod_{\substack{\omega\in\Lambda\\ \omega\neq 0}}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

Then

$$\frac{d}{dz}\log\sigma(z;\Lambda) = \zeta(z;\Lambda) \quad\text{and}\quad \frac{d^2}{dz^2}\log(\sigma(z;\Lambda)) = -\wp(z;\Lambda).$$

$$\wp(z)-\wp(a) = \frac{-\sigma(z+a)\sigma(z-a)}{\sigma(z)^2\sigma(a)^2} \quad\text{and}\quad \wp'(z) = \frac{-\sigma(2z)}{\sigma(z)^4}.$$