## Review

$$\mathcal{M} = \left\{ \Omega = \begin{pmatrix} -\omega_1- \\ -\omega_2- \\ \vdots \\ -\omega_g- \end{pmatrix} \in M_{2g\times g}(\mathbb{C}) \,\middle|\, \det(\Omega, \overline{\Omega}) \neq 0 \right\}.$$

and

$$\left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of complex tori} \end{array} \right\} \xleftrightarrow{\sim} GL_{2g}(\mathbb{Z}) \backslash \mathcal{M} / GL_g(\mathbb{C})$$

$$\mathbb{C}^g / \Lambda(\Omega) \longleftarrow\!\!\shortmid \ \Omega$$

where

$$\Lambda(\Omega) = \mathbb{Z}\text{-span}\{\omega_1, \omega_2, \ldots, \omega_{2g}\}$$

($\Lambda(\Omega)$ is rank $2g$ if $\det(\Omega, \overline{\Omega}) \neq 0$).

If $\Omega \in M_{2g\times g}(\mathbb{C})$ then the orbit

$GL_{2g}(\mathbb{Z})\,\Omega\, GL_g(\mathbb{C})$ has a representative $\begin{pmatrix} \tau \\ 1 \end{pmatrix}$

with $\tau \in M_{g\times g}(\mathbb{C})$ and $1$ the $g\times g$ identity matrix.

If $M \in GL_{2g}(\mathbb{Z})$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

$$M \begin{pmatrix} \tau \\ 1 \end{pmatrix} (C\tau + D)^{-1} = \begin{pmatrix} (A\tau+B)(C\tau+D)^{-1} \\ 1 \end{pmatrix}$$

An <u>abelian variety</u> is a complex torus $\mathbb{C}^g/\Lambda$ which imbeds in projective space.

An <u>elliptic curve</u> is an abelian variety with $g=1$.

## The case $g=1$

$$\mathfrak{S}_1 = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) \in \mathbb{R}_{>0}\}$$

$$\Gamma(1) = \frac{SL_2(\mathbb{Z})}{\{\pm 1\}} = PSL_2(\mathbb{Z}).$$

The moduli space of elliptic curves is

$$X(1) = \Gamma(1)\backslash\mathfrak{S}_1 \overset{\sim}{\longleftrightarrow} \frac{\{\text{lattices in } \mathbb{C}\}}{\mathbb{C}^\times} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{l}\text{isom. classes}\\ \text{of elliptic curves}\end{array}\right\}$$

$$\tau \longmapsto \mathbb{Z}\text{-span}\{1, \tau\} \longmapsto \frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}}.$$

where $\Gamma(1)$ acts on $\mathfrak{S}_1$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (A\tau + B)(C\tau + D)^{-1}.$$

Then

$$\mathfrak{S}_1 = SL_2(\mathbb{R}) / SO_2(\mathbb{R})$$

so that

$$\Gamma(1)\backslash\mathfrak{S}_1 = \Gamma(1)\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R}) = SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R}).$$

<u>The general g case</u>: [Igusa p. 24 and Chi I Theorem 5].

The <u>Siegel upper half space</u> is

$$\mathfrak{S}_g = \{\tau \in M_g(\mathbb{C}) \mid \tau = \tau^t \text{ and } \operatorname{Im}(q) \text{ is pos. definite}\}$$

Then

(a) $$\mathfrak{S}_g \simeq Sp_{2g}(\mathbb{R}) / Sp_{2g}(\mathbb{R}) \cap O_{2g}(\mathbb{R})$$

(b) $K = Sp_{2g}(\mathbb{R}) \cap O_{2g}(\mathbb{R})$ is a maximal compact subgroup of $Sp_{2g}(\mathbb{R})$.

(c) All maximal compact subgroups of $Sp_{2g}(\mathbb{R})$ are conjugate.

Let $d_1, \dots, d_g \in \mathbb{Z}_{>0}$ with $d_i \mid d_{i+1}$. Let

$$\Delta = \left(\begin{array}{c|c} 0 & \begin{matrix} d_1 & & \\ & \ddots & \\ & & d_g \end{matrix} \\ \hline \begin{matrix} -d_1 & & \\ & \ddots & \\ & & -d_g \end{matrix} & 0 \end{array}\right)$$

$$Sp(\Delta, \mathbb{Z}) = \{M \in GL_{2g}(\mathbb{Z}) \mid M \Delta M^t = \Delta\}$$

acting on $\mathfrak{S}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tau = (A\tau + B\Delta)(C\tau + D\Delta)^{-1}.$$

Then

$$\left\{\begin{array}{c}\text{Isom. classes}\\ \text{of abelian varieties}\\ \text{with } \Delta\text{-polarization}\end{array}\right\} \overset{\sim}{\longleftrightarrow} Sp(\Delta,\mathbb{Z})\backslash \mathfrak{S}_g \overset{\sim}{\longleftrightarrow} Sp(\Delta,\mathbb{Z})\backslash Sp_{2g}(\mathbb{R}) / Sp_{2g}(\mathbb{R}) \cap O_{2g}(\mathbb{R})$$

$$\left(\mathbb{C}^g/\Lambda(\tau), \mathcal{L}\right) \longleftarrow \tau$$

An <u>abelian variety</u> is a complex variety $\mathbb{C}^g/\Lambda$ which can be embedded into projective space $\mathbb{P}^n(\mathbb{C})$

A <u>polarized abelian variety</u> is a pair $(\mathbb{C}^g/\Lambda, \mathcal{L})$ where $\mathbb{C}^g/\Lambda$ is an abelian variety and $\mathcal{L}$ is an ample line bundle on $\mathbb{C}^g/\Lambda$.

<u>Kodaira Embedding Theorem</u> Let $X$ be a compact complex manifold with a Hermitian metric $H: T_X \times T_X \to \mathbb{C}$ such that $\omega_H \in H^2(X,\mathbb{Z})$, the class corresponding to $H$, is closed and integral. Then there is an ample line bundle $\mathcal{L}$ on $X$ with $c_1(\mathcal{L}) = \omega_H$ and

$$X \longrightarrow \mathbb{P}(H^0(X, \mathcal{L}^{\otimes n}))$$

$$x \longmapsto H_x = \{s \in H^0(X, \mathcal{L}^{\otimes n}) \mid s(x) = 0\}$$

is an embedding for suitably large $n$.
(see [Ha, Theorem 5.2.19] and [Serre, GAGA]).

# Theta functions again

$\mathfrak{G}_1 = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) \in \mathbb{R}_{>0}\}$ and $q = e^{i\pi\tau}$

The <u>Jacobi</u> <u>theta</u> <u>function</u> $\theta : \mathbb{C} \times \mathfrak{G}_1 \to \mathbb{C}$ is

$$\theta(z,q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} e^{2\pi i n z}$$

(see [AAR (10.7.4)]
[TF App. I (I.1)])

$\mathfrak{G}_g = \{\tau \in M_{g\times g}(\mathbb{C}) \mid \tau = \tau^t$ and $\mathrm{Im}(\tau)$ is pos. def.$\}$

The <u>Riemann</u> <u>theta function</u> $\theta : \mathbb{C}^g \times \mathfrak{G}_g \to \mathbb{C}$ is

$$\theta(z,\tau) = \sum_{\ell \in \mathbb{Z}^g} e^{2\pi i (\frac{1}{2}\ell\tau\ell^t + \ell z^t)}$$

(see [SU (2.34)])

Kac-Peterson define the <u>Riemann</u> <u>theta</u> <u>function</u> as

$$\theta(\tau,z,t) = e^{-2\pi i t} \sum_{\gamma \in L} e^{2\pi i (\frac{1}{2}\tau\langle\gamma,\gamma\rangle - \langle\gamma,z\rangle)} \quad \text{for } (\tau,z,t) \in \mathfrak{G}_1 \times \mathbb{C}^g \times \mathbb{C}.$$

(see [KP, right after Prop. 3.1])

Let $\mathcal{L}$ be an ample line bundle on $\mathbb{C}^g/\Lambda(\Omega)$.

A <u>Riemann</u> <u>theta function</u> is a (global) section of $\mathcal{L}^{\otimes d}$ with $d \in \mathbb{Z}_{>0}$ (see [Ha, Prop. 5.2.33] and [SU Theorem 2.13]).

A <u>theta function</u> on $\mathbb{C}^g$ relative to $\Lambda$ is a holomorphic function $\theta : \mathbb{C}^g \to \mathbb{C}$ such that if $z \in \mathbb{C}^g$ and $\xi \in \Lambda$ then

$$\theta(z+\xi) = \theta(z)\, e^{2\pi i (Q_\xi(z) + c_\xi)},$$

where

$Q_\xi : \mathbb{C}^g \to \mathbb{C}$ is $\mathbb{C}$-linear and $c_\xi \in \mathbb{C}$, for $\xi \in \Lambda$ (see [Ig p.60] and [HS, A.5.2, p 97]).

Abelian varieties and Thetafunctions

<u>The Heisenberg group $A(X)$</u> from Igusa. 18.03.2014

$X$ a commutative locally compact group.

$X^* = Hom(X, U_1(\mathbb{C}))$

Our favourite example is $X = \mathbb{R}^n$ and $X^*$ the dual of $X$.

<u>[Ig, Theorem 3].</u>

(a) $A(X) = X \times X^* \times U_1(\mathbb{C})$ with

$$(u_1, u_1^*, t_1)(u_2, u_2^*, t_2) = (u_1 + u_2,\ u_1^* + u_2^*,\ \langle u_1, u_2^* \rangle t_1 t_2)$$

(b) $A(X)$ acts on ~~Aut~~ $L^2(X)$ by

$$((u, u^*, t)f)(x) = t \langle x, u^* \rangle f(x+u).$$

This action of $A(X)$ on $L^2(X)$ is the <u>Schrödinger representation</u>.

The centralizer of $A(X)$ in $Aut(L^2(X))$ is $U_1(\mathbb{C})$

The normalizer of $A(X)$ in $Aut(L^2(X))$ is $B(X)$.

Then $\quad B(X) = Sp(X) \ltimes (X \times X^* \times U_1(\mathbb{C}))$

so that

$$Sp(X) = B(X)/A(X) \quad \text{and} \quad G_g = Sp_{2g}(\mathbb{R})/K = \frac{B(\mathbb{R}^n)/A(\mathbb{R}^n)}{\hat{K}/A(\mathbb{R}^n)} = \frac{B(\mathbb{R}^n)}{\hat{K}}$$

where

$$Sp(X) = \{\sigma \in GL(X \times X^*) \mid \langle (x,x^*),(y,y^*) \rangle = \langle \sigma(x,x^*), \sigma(y,y^*) \rangle$$

with $\quad \langle (x,x^*),(y,y^*) \rangle = \langle x, y^* \rangle \langle -y, x^* \rangle$

# Review and elliptic curves

The Weierstrass $\wp$-function is

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

where $\Lambda = \mathbb{Z}\text{span}\{\omega_1, \omega_2\}$ in $\mathbb{C}$.

The elliptic functions are meromorphic $f: \mathbb{C}\to\mathbb{C}$ such that

if $w\in\Lambda$ and $z\in\mathbb{C}$ then $f(z+w) = f(z)$

Theorem The ring of elliptic functions is

$$(\mathcal{O}_{\mathbb{C}})^{\Lambda} = \mathbb{C}(\wp(z), \wp'(z)) \quad \text{where } \wp'(z) = \frac{d}{dz}\wp(z).$$

Let $k\in\mathbb{Z}_{\geq 2}$. The Eisenstein series of weight $2k$ is

$$G_{2k}(\Lambda) = \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \frac{1}{\omega^{2k}}$$

Let $G_k(\Lambda) = 0$ for $k\in\mathbb{Z}_{\leq 0}$ and $k\in\mathbb{Z}_{odd}$. Let

$$g_2 = 60\,G_4 \quad \text{and} \quad g_3 = 140\,G_6.$$

The space of modular forms is

$$\bigoplus_{k\in\mathbb{Z}_{\geq 0}} M_{2k}, \quad \text{where } M_{2k} = \mathrm{span}\left\{ G_4^a G_6^b \;\middle|\; \begin{array}{l} a,b\in\mathbb{Z}_{\geq 0} \\ 4a+6b=2k \end{array} \right\}$$

Proposition

(a) $\wp(z) = z^{-2} + \sum_{k\in\mathbb{Z}_{>0}} (2k+1) G_{2k+2}\, z^{2k}$

(b) $(\wp'(z))^2 = 4\wp(z) - g_2\wp(z) - g_3$

(c)
$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{\sim} & E \subseteq \mathbb{P}^2(\mathbb{C}) \\ z & \longmapsto & [\wp(z), \wp'(z), 1] \end{array}$$

where $E = \{[x,y,1]\in\mathbb{P}^2(\mathbb{C}) \mid y^2 = 4x^3 - g_2x - g_3\}$

Proposition [AEC II, Ch. I Prop. 4.4]. Let

$\Gamma(1) = \frac{SL_2(\mathbb{Z})}{\{\pm 1\}} = PSL_2(\mathbb{Z})$ and $\mathfrak{G}_1 = \{\tau\in\mathbb{C} \mid \mathrm{Im}(\tau)\in\mathbb{R}_{>0}\}$

Then

$$\begin{array}{ccccccc} \left\{\begin{array}{c}\text{isom classes} \\ \text{of elliptic curves}\end{array}\right\} & \xleftarrow{\sim} & \dfrac{\{\text{lattices } \Lambda \text{ in } \mathbb{C}\}}{\mathbb{C}^\times} & \xleftarrow{\sim} & \Gamma(1)\backslash\mathfrak{G}_1 & \xrightarrow{\sim} & \mathbb{C} \\ \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} & \longleftarrow & \mathbb{Z}\mathrm{span}\{1,\tau\} & \longleftarrow & \tau & \longmapsto & j(\tau) \end{array}$$

where

$$j(\tau) = 1728\frac{g_2(\tau)^3}{\Delta(\tau)} \quad \text{with} \quad \Delta(\tau) = g_2 - 27g_3.$$

# Lattices in $\mathbb{C}^g$

A matrix

$$\Omega = \begin{pmatrix} - \omega_1 - \\ - \omega_2 - \\ \vdots \\ - \omega_{2g} - \end{pmatrix} \in M_{2g \times g}(\mathbb{C})$$

determines a lattice

$$\Lambda(\Omega) = \mathbb{Z}\text{span}\{\omega_1, \omega_2, \ldots, \omega_{2g}\} \quad \text{in } \mathbb{C}^g.$$

The lattice has rank $2g$ if and only if

$$\det \begin{pmatrix} - \omega_1 - & - \overline{\omega_1} - \\ - \omega_2 - & - \overline{\omega_2} - \\ \vdots & \\ - \omega_{2g} - & - \overline{\omega_{2g}} - \end{pmatrix} = \det(\Omega, \overline{\Omega}) \neq 0$$

If $\Lambda(\Omega)$ has rank $2g$ then

$\mathbb{C}^g / \Lambda(\Omega)$ is a compact complex $g$-manifold

and $\mathbb{C}^g / \Lambda(\Omega)$ is a <u>complex torus</u>.

If $M \in GL_{2g}(\mathbb{Z})$ then $\Lambda(\Omega) = \Lambda(M\Omega)$.

If $G \in GL_g(\mathbb{C})$ then $\mathbb{C}^g / \Lambda(\Omega) \simeq \mathbb{C}^g / \Lambda(\Omega G)$

Let

$$\mathcal{M} = \{\Omega \in M_{2g \times g}(\mathbb{C}) \mid \det(\Omega, \overline{\Omega}) \neq 0\}$$

Then

$$GL_{2g}(\mathbb{Z}) \backslash \mathcal{M} / GL_g(\mathbb{C}) \xleftrightarrow{\sim} \left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of complex tori} \end{array} \right\}$$

$$\Omega \longmapsto \mathbb{C}^g / \Lambda(\Omega).$$

## The case $g=1$

$$\Omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ and } \Lambda = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\}$$

$$GL_g(\mathbb{C}) = GL_1(\mathbb{C}) = \mathbb{C}^\times \text{ and } GL_{2g}(\mathbb{Z}) = GL_2(\mathbb{Z}).$$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \left(\frac{1}{\omega_2}\right) = \begin{pmatrix} \frac{\omega_1}{\omega_2} \\ 1 \end{pmatrix} = \begin{pmatrix} x+iy \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x+iy \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ x+iy \end{pmatrix}$$

and

$$\begin{pmatrix} 1 \\ x+iy \end{pmatrix} \left(\frac{1}{x+iy}\right) = \begin{pmatrix} \frac{1}{x+iy} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{x-iy}{x^2+y^2} \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{x-iy}{x^2+y^2} \\ 1 \end{pmatrix} (x^2+y^2) = \begin{pmatrix} x-iy \\ 1 \end{pmatrix}$$

If

$$\mathcal{M} = \left\{ \Omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \;\middle|\; \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \text{ is rank } 2 \right\}$$

and $\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in \mathcal{M}$ then $\omega_1 \neq 0$ and $\omega_2 \neq 0$ and

$$\frac{\mathbb{C}}{\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2} \simeq \frac{\mathbb{C}}{\mathbb{Z} + \mathbb{Z}(x+iy)} \simeq \frac{\mathbb{C}}{\mathbb{Z} + \mathbb{Z}(x-iy)}$$

and $\mathbb{Z} + \mathbb{Z}(x+iy)$ is rank 2 if $y \neq 0$.

So let

$$\begin{aligned} G_1 &= \{ \tau = x+iy \in \mathbb{C} \mid y \in \mathbb{R}_{>0} \} \\ &= \{ \tau \in \mathbb{C} \mid \mathrm{Im}(\tau) \in \mathbb{R}_{>0} \}. \end{aligned}$$

Then

$$GL_2(\mathbb{Z}) G_1 GL_1(\mathbb{C}) = S_2 G_1 \mathbb{C}^\times = \mathcal{M}.$$

If

$$\Omega = \begin{pmatrix} \text{———} \; w_1 \; \text{———} \\ \text{———} \; w_2 \; \text{———} \\ \vdots \\ \text{———} \; w_{2g} \; \text{———} \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \in M_{2g \times g}(\mathbb{C})$$

Then let $\sigma \in S_{2g}$ (the symmetric group $S_{2g} \subseteq GL_{2g}(\mathbb{Z})$)

so that

$$\sigma\Omega = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \quad \text{with} \quad \det(\tau_2) \neq 0.$$

Then $\sigma\Omega\tau_2^{-1} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}\tau_2^{-1} = \begin{pmatrix} \tau_1\tau_2^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} \tau \\ 1 \end{pmatrix}$

Then, if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2g}(\mathbb{Z})$ then

$$M\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} A\tau+B \\ C\tau+D \end{pmatrix}$$

and

$$M\begin{pmatrix} \tau \\ 1 \end{pmatrix}(C\tau+D)^{-1} = \begin{pmatrix} A\tau+B \\ C\tau+D \end{pmatrix}(C\tau+D)^{-1} = \begin{pmatrix} (A\tau+B)(C\tau+D)^{-1} \\ 1 \end{pmatrix}$$