Let (W,S) with $S=\{s_1,\ldots,s_\ell\}$ be a Coxeter group:

$$s_i^2=1 \quad\text{and}\quad \underbrace{s_is_js_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{s_js_is_j\cdots}_{m_{ij}\text{ factors}} \quad \text{for } i\neq j$$

Let $V=\mathfrak{h}_{\mathbb{R}}^*$ be the reflection representation,

$$s_i:\mathfrak{h}_{\mathbb{R}}^*\longrightarrow\mathfrak{h}_{\mathbb{R}}^* \quad\text{given by}\quad s_i\lambda=\lambda-\langle\lambda,\alpha_i^\vee\rangle\alpha_i,$$

R^+ an index set for reflections in W, and

$$\mathfrak{h}^{\alpha^\vee}=\mathfrak{h}^{s_\alpha}=\{x\in\mathfrak{h}_{\mathbb{R}}^*\mid s_\alpha x=x\}=\{x\in\mathfrak{h}_{\mathbb{R}}^*\mid\langle x,\alpha^\vee\rangle=0\}$$

The <u>chamber</u> is $C=\{x\in\mathfrak{h}_{\mathbb{R}}^*\mid\langle x,\alpha^\vee\rangle>0 \text{ for } \alpha\in R^+\}$ and the <u>Tits cone</u> is

$$\mathcal{I}=\bigcup_{w\in W}w(\bar{C}) \quad\text{and}\quad \mathcal{I}^\circ \text{ is the interior of } \mathcal{I}.$$

The affine Artin group of W

$$\tilde{Y} = \mathfrak{h}_{\mathbb{R}}^* + i I^\circ - \bigcup_{\substack{\alpha \in R^+ \\ m \in \mathbb{Z}}} \mathfrak{h}^{\alpha^\vee, m} + i \mathfrak{h}^{\alpha^\vee}$$

where
$$\mathfrak{h}^{\alpha^\vee, m} = \{ x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha^\vee \rangle = m \}$$

and
$$\dim(\mathfrak{h}_{\mathbb{R}}^*) \geq \ell + \operatorname{corank}(N) \quad \text{where } N = (\alpha_i^\vee(\alpha_j))_{1 \leq i,j \leq \ell}.$$

Let
$$Q = \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i \quad \text{and} \quad \tilde{W} = W \ltimes Q = \{ w t_\mu \mid w \in W, \mu \in Q \}$$

with $t_\mu t_\nu = t_{\mu+\nu}$ and $w t_\mu = t_{w\mu} w$. Then $\tilde{W}$ acts on $\tilde{Y}$ by

$$w(x+iy) = wx + i(wy) \quad \text{and} \quad t_\mu(x+iy) = (\mu+x) + iy.$$

and the basepoint is .....

<u>Theorem</u> $\pi_1(\tilde{Y}/\tilde{W})$ is generated by $T_1, \ldots, T_\ell$ and $X^{\alpha_1}, \ldots, X^{\alpha_\ell}$ with relations

$$X^{\alpha_i} X^{\alpha_j} = X^{\alpha_j} X^{\alpha_i} \quad \text{and} \quad \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}} \quad \text{for } i \neq j.$$

$$T_i X^{\alpha_j} = X^{\alpha_j} X^{r\alpha_i} T_i X^{-r\alpha_i}, \quad \text{if } -\langle \alpha_i^\vee, \alpha_j \rangle = 2r,$$
$$T_i X^{\alpha_j} = X^{\alpha_j} X^{(r+1)\alpha_i} T_i X^{-r\alpha_i}, \quad \text{if } -\langle \alpha_i^\vee, \alpha_j \rangle = 2r+1.$$

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3 pole braids

(3.5)

Examples $N=(2)$ so that $W=\langle s_1 \mid s_1^2=1\rangle$.

In the first case $\dim(\mathfrak{h}_{\mathbb{R}}^*)=1$ and

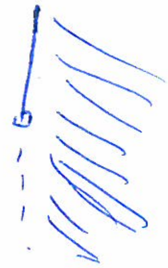
$\mathfrak{h}_{\mathbb{R}}^* = \mathrm{span}\{\alpha_1\}$ with $s_1\alpha_1=-\alpha_1$

and

$$Y = \mathfrak{h}_{\mathbb{R}}^* + i\mathfrak{h}_{\mathbb{R}}^* - (\mathfrak{h}^{\alpha_1^\vee} + i\mathfrak{h}^{\alpha_1^\vee}) = \mathbb{C}\alpha_1 - 0 \simeq \mathbb{C}^\times$$

since $\mathfrak{h}^{\alpha_1^\vee} = \{x \in \mathbb{R} \mid \langle \alpha_1^\vee, x\rangle = 0\} = 0$.

So $Y = \mathbb{C}-\{0\}$ and $Y/W =$

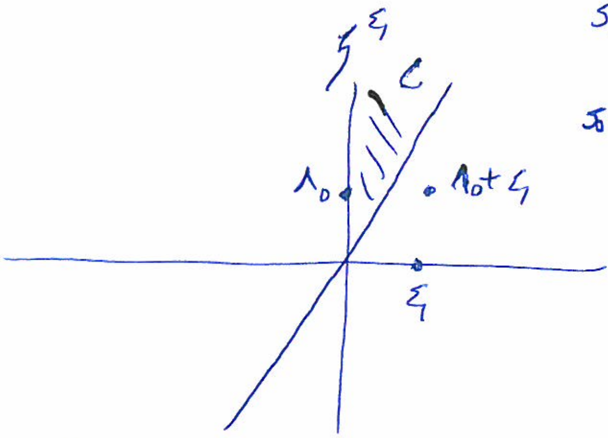


<u>Example</u> $N = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ so that $W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1 \rangle$.

In the first case $\dim(\mathfrak{h}_{\mathbb{R}}^*) = \ell = 2$ and

$\mathfrak{h}_{\mathbb{R}}^* = \mathrm{span}\{\Lambda_0, \varepsilon_1\}$ with $s_0\Lambda_0 = \Lambda_0 + \varepsilon_1$ $\quad s_0\varepsilon_1 = -\varepsilon_1$

$s_1\Lambda_0 = \Lambda_0$ $\quad s_1\varepsilon_1 = -\varepsilon_1$



$s_0 s_1(x_1\varepsilon_1 + x_2\Lambda_0)$
$= s_0(-x_1\varepsilon_1 + x_2\Lambda_0)$
$= x_1\varepsilon_1 + x_2(\Lambda_0 + \varepsilon_1) = x_1\varepsilon_1 + x_2\Lambda_0 + x_2\varepsilon_1$

Then $\alpha_1^\vee = \varepsilon_1$ and $\alpha_0^\vee = \frac{1}{2}\delta - \varepsilon_1$ so that

$\alpha_0^\vee(\Lambda_0 + \frac{1}{2}\varepsilon_1) = \langle \frac{1}{2}\delta - \varepsilon_1, \Lambda_0 + \frac{1}{2}\varepsilon_1 \rangle = \frac{1}{2} - \frac{1}{2} = 0$

$\alpha_1^\vee(\Lambda_0) = \langle \varepsilon_1, \Lambda_0 \rangle = 0.$

So $\mathfrak{h}_{\mathbb{R}}^* = \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid x_1, x_2 \in \mathbb{R} \}$ and

$\mathfrak{h}^{\alpha_1^\vee} = \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid \langle \alpha_1^\vee, x_1\varepsilon_1 + x_2\Lambda_0 \rangle = 0 \}$
$= \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid x_1 = 0 \} = \mathbb{R}\Lambda_0$

$\mathfrak{h}^{\alpha_0^\vee} = \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid \langle \alpha_0^\vee, x_1\varepsilon_1 + x_2\Lambda_0 \rangle = 0 \}$
$= \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid \langle \frac{1}{2}\delta - \varepsilon_1, x_1\varepsilon_1 + x_2\Lambda_0 \rangle = 0 \}$
$= \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid \frac{1}{2}x_2 - x_1 = 0 \}$
$= \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid x_2 = 2x_1 \}.$

The coroots are $W \cdot \{\alpha_0^\vee, \alpha_1^\vee\}$

The chamber is $C = \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid x_1 > 0, x_2 > 2x_1 \}$

The Tits Cone is $I = \bigcup_{w \in W} w\overline{C} = \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid x_2 > 0 \} = \mathbb{R}\varepsilon_1 + \mathbb{R}_{>0}\Lambda_0.$

So $\quad \mathcal{I}^\circ = \mathbb{R}\varepsilon_1 + \mathbb{R}_{>0}\Lambda_0.$

So

$$Y = \mathfrak{z}^*_{\mathbb{R}} + i\,\mathfrak{z}^*_{>0} - \bigcup_{\alpha\in R^+} \mathfrak{z}^{\alpha^\vee} + i\,\mathfrak{z}^{\alpha^\vee}.$$

$$= (\mathbb{R}\varepsilon_1 + \mathbb{R}_{>0}\Lambda_0) - \bigcup_{\alpha\in R^+} \mathfrak{z}^{\alpha^\vee} + i\,\mathfrak{z}^{\alpha^\vee}.$$

Use

$\alpha_0^\vee = -\varepsilon_1 + \frac{1}{2}\delta$ and $\alpha_1^\vee = \varepsilon_1$ and

$s_0\alpha_0^\vee = s_0(-\varepsilon_1 + \frac{1}{2}\delta) = -\delta + \varepsilon_1 + \frac{1}{2}\delta = \varepsilon_1 - \frac{1}{2}\delta,$

$s_0\varepsilon_1 = \delta - \varepsilon_1$ and $s_1\varepsilon_1 = -\varepsilon_1,\quad s_0\delta = \delta,\quad s_1\delta = \delta.$

So $\quad W\cdot\{\alpha_0^\vee, \alpha_1^\vee\} = W\cdot\{-\varepsilon_1 + \frac{1}{2}\delta, \varepsilon_1\} = \{\pm\varepsilon_1 + \frac{1}{2}\delta + k\delta, \pm\varepsilon_1 + k\delta \mid k\in\mathbb{Z}\}$

$= \{\pm\varepsilon_1 + k\delta,\ \pm\varepsilon_1 + (k+\frac{1}{2})\delta \mid k\in\mathbb{Z}\}$

Then

$$\mathfrak{z}^{\pm\varepsilon_1 + k\delta} = \{x_1\varepsilon_1 + x_2\Lambda_0 \mid \langle \pm\varepsilon_1 + k\delta, x_1\varepsilon_1 + x_2\Lambda_0\rangle = 0\}$$

$$= \{x_1\varepsilon_1 + x_2\Lambda_0 \mid \pm x_1 + kx_2 = 0\}$$

$$= \{x_1\varepsilon_1 + x_2\Lambda_0 \mid x_2 = \pm\tfrac{1}{k}x_1\} = \{x_1\varepsilon_1 + x_2\Lambda_0 \mid x_2\tfrac{2k}{2} = \pm x_1\}$$

$$\mathfrak{z}^{\pm\varepsilon_1 + \frac{2k+1}{2}\delta} = \{x_1\cancel{\varepsilon_1} + x_2\Lambda_0 \mid x_2 = \tfrac{\pm 2}{2k+1}x_1\}$$

$$= \{x_1\varepsilon_1 + x_2\Lambda_0 \mid x_2\tfrac{(2k+1)}{2} = \pm x_1\}$$

So

$$Y = \cancel{\mathbb{R}\varepsilon_1 + i\mathbb{R}_{>0}\Lambda_0}$$

$$\mathbb{C}\varepsilon_1 + \mathbb{C}_1\Lambda_0 - \bigcup_{m\in\mathbb{Z}} \{z_1\varepsilon_1 + z_2\Lambda_0 \mid \tfrac{m}{2}z_2 = z_1\}.$$

Type $A_1^{(1)}$ $\qquad N = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

So $\quad s_0\alpha_1 = \alpha_1 - \langle \alpha_1, \alpha_0^\vee \rangle \alpha_0 = \alpha_1 + 2\alpha_0 \qquad s_0\alpha_0 = -\alpha_0$

$\quad s_1\alpha_1 = -\alpha_1 \qquad s_1\alpha_0 = \alpha_0 + 2\alpha_1$.

Now $\alpha_0 = -\alpha_1 + \delta$. $\quad$ So $\delta = \alpha_0 + \alpha_1$.

So $\quad s_0\delta = s_0(\alpha_0 + \alpha_1) = -\alpha_0 + \alpha_1 + 2\alpha_0 = \alpha_1 + \alpha_0 = \delta$. $\qquad s_0(\alpha_1) = \delta + \alpha_0 = 2\delta - \alpha_1$

$\quad s_1\delta = s_1(\alpha_0 + \alpha_1) = \alpha_0 + 2\alpha_1 - \alpha_1 = \alpha_0 + \alpha_1 = \delta$.

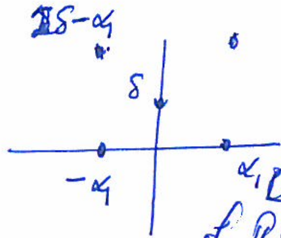
and

$s_0 s_1 \alpha_1 = s_0(-\alpha_1) = -\alpha_1 - 2\alpha_0 = -\delta - \alpha_0 = -\delta + \alpha_1 - \delta = \alpha_1 - 2\delta$.

Since

$$s_i(\lambda + \mu) = \lambda + \mu - \langle \lambda + \mu, \alpha_i^\vee \rangle \alpha_i$$
$$= \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i + \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i.$$

so that s_i is a linear transformation.



This is my level 0 piece of $\mathbb{R}\delta + \mathbb{R}\alpha_1 + \mathbb{R}\Lambda_0$

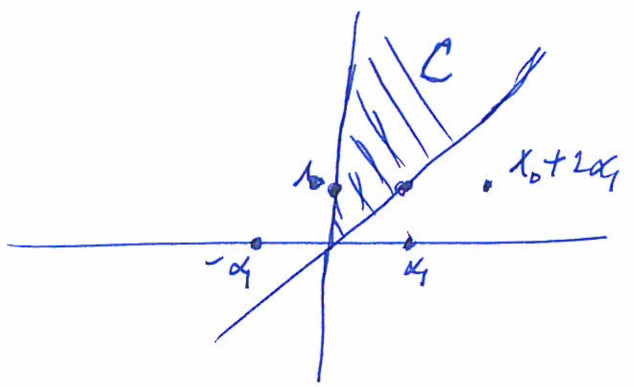
So, in the basis $\{\delta, \alpha_1\}$

$$s_0 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \qquad s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The dual basis is $\{\Lambda_0, \alpha_1\}$ and in this basis

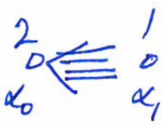
$s_0 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ so $\quad s_0\Lambda_0 = \Lambda_0 + 2\alpha_1 \qquad s_0\alpha_1 = -\alpha_1$

$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so $\quad s_1\Lambda_0 = \Lambda_0 \qquad s_1\alpha_1 = -\alpha_1$



$s_0(\Lambda_0 + \alpha_1) = \Lambda_0 + 2\alpha_1 - \alpha_1$
$= \Lambda_0 + \alpha_1$.

Type $A_2^{(2)}$ [Kac Ex 6.4]



$$A=\begin{pmatrix}2 & -4\\ -1 & 2\end{pmatrix}$$

So

$$s_0\alpha_0=-\alpha_0 \qquad s_1\alpha_0=\alpha_0-\langle\alpha_1^\vee,\alpha_0\rangle\alpha_1 = \alpha_0+\alpha_1$$

$$s_0\alpha_1=\alpha_1-\langle\alpha_0^\vee,\alpha_1\rangle\alpha_0 = \alpha_1+4\alpha_0 \qquad s_1\alpha_1=-\alpha_1$$

and $\delta=a_0\alpha_0+a_1\alpha_1=2\alpha_0+\alpha_1$

$$\begin{pmatrix}2 & -4\\ -1 & 2\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

So

$$s_0(\delta)=s_0(2\alpha_0+\alpha_1)=2(-\alpha_0)+\alpha_1+4\alpha_0=\alpha_1+2\alpha_0=\delta.$$

$$s_1\delta=s_1(2\alpha_0+\alpha_1)=2(\alpha_0+\alpha_1)-\alpha_1=2\alpha_0+\alpha_1=\delta.$$

So

$$\alpha_1=\alpha_1 \qquad \text{and} \qquad \alpha_1=\alpha_1$$

$$\delta=2\alpha_0+\alpha_1 \qquad \qquad \alpha_0=\frac{-1}{2}\alpha_1+\frac{1}{2}\delta=-\frac{1}{2}\alpha_1+\frac{1}{2}\delta.$$

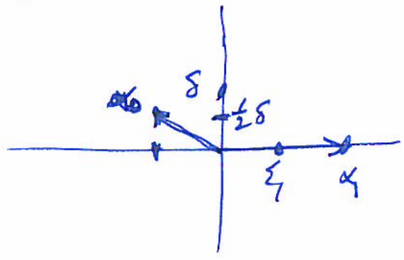
So, if $2\varepsilon_1=\alpha_1$ then $\alpha_0=-\varepsilon_1+\frac{1}{2}\delta$.

and $s_1\varepsilon_1=-\varepsilon_1$ and

$$\begin{aligned}s_0\varepsilon_1&=\tfrac{1}{2}s_0\alpha_1=\tfrac{1}{2}(\alpha_1+4\alpha_0)\\&=\tfrac{1}{2}(\delta+2\alpha_0)=\tfrac{1}{2}\delta+\alpha_0=\tfrac{1}{2}\delta-\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\delta\\&=\delta-\varepsilon_1.\end{aligned}$$

So, in the basis $\{\delta,\varepsilon_1\}$

$$s_0=\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix} \quad \text{and} \quad s_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$



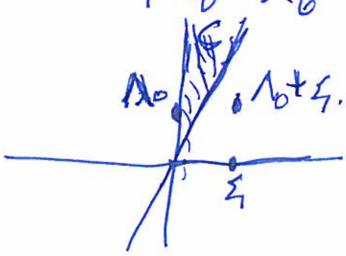
The dual basis is $\{\Lambda_0,\varepsilon_1\}$ and in this basis

$$s_0=\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix} \qquad s_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

So

$$s_0\Lambda_0=\Lambda_0+\varepsilon_1 \qquad s_0\varepsilon_1=-\varepsilon_1$$

$$s_1\Lambda_0=\Lambda_0 \qquad s_1\varepsilon_1=-\varepsilon_1$$



$$s_0(\Lambda_0+\tfrac{1}{2}\varepsilon_1)=\Lambda_0+\varepsilon_1-\tfrac{1}{2}\varepsilon_1=\Lambda_0+\tfrac{1}{2}\varepsilon_1$$