Conference Talk
CRM Montreal 24/04/2014 ①

Are 3-pole braids elliptic?

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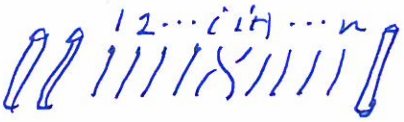
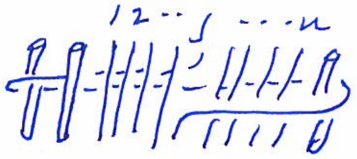
The observation: The double affine Artin group $\tilde{\mathcal{B}}_n$ of type $(C_n^\vee, C_n)$ is braids with n strands and 3-poles.

$T_i =$ 

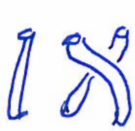
for $i = 1, 2, \dots, n-1$

$T_0 =$  and $T_n =$  and

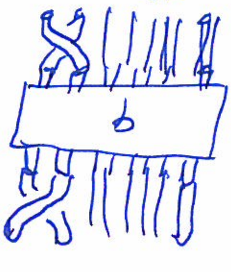
$X^{\varepsilon_j} =$ 

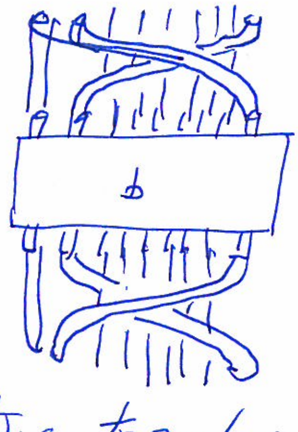
for $j = 1, 2, \dots, n$.

Then

$\mathcal{B}_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle = \langle \sigma_1 =$  , $\sigma_2 =$  $\rangle$

acts on $\tilde{\mathcal{B}}_n$

$\sigma_1 b =$ 

and $\sigma_2 b =$ 

Question 1: Is this obvious from the topological construction of the DAArt? (Ion 2003, Cherednik 1992, van der Lek 1983, Birman 1969)

Question 2: Is the $\mathcal{B}_3$-action the same as Ion–Sahi 2003?

Question 3: Is the $\mathcal{B}_3$-action coming from the $SL_2(\mathbb{Z})$ action on the upper half plane $\mathbb{R} + i\mathbb{R}_{>0}$?

(moduli space of elliptic curves).

Topological construction vander Lek / Looijenga / Kac–Peterson.
1983 / 1976 / 1984

Affine Kac Moody Lie algebra $\quad \mathfrak{g} = \mathbb{C}K \oplus \mathring{\mathfrak{g}}[t,t^{-1}] \oplus \mathbb{C}d.$

Cartan subalgebra $\quad \mathfrak{h} = \mathbb{C}K \oplus \mathring{\mathfrak{h}} \oplus \mathbb{C}d.$

$$\mathfrak{h} \ \underset{\nu}{\overset{\wr}{\downarrow}}$$

Weights $\quad \mathfrak{h}^* = \mathbb{C}\delta \oplus \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\delta.$

Complexified Tits cone $\quad \mathfrak{h}_{>0} = \mathbb{C}K \oplus \mathring{\mathfrak{h}} \oplus (\mathbb{R} + i\mathbb{R}_{>0})d$

is where Weyl characters converge (irred. integrable h.w. ~~$\mathfrak{g}$~~-modules).

The affine Weyl group
$$W = \left\{ \left(\begin{array}{c|c|c} 1 & -\beta & -\frac{1}{2}(\beta|\beta) \\ \hline 0 & w & \beta \\ \hline 0 & 0 & 1 \end{array} \right) \;\middle|\; \begin{array}{l} w \in W_0 \\ \beta \in \mathring{\mathfrak{h}}_{\mathbb{Z}} \end{array} \right\}$$
W_0 = finite Weyl group

$\mathring{\mathfrak{h}}_{\mathbb{Z}}$ = coroot lattice

The double affine Weyl group DAWG
$$\tilde{W} = W \ltimes \mathfrak{h}_{\mathbb{Z}} = \left\{ \left(\begin{array}{c|c|c} 1 & \mu & k \\ \hline 0 & w & \nu \\ \hline 0 & 0 & 1 \end{array} \right) \;\middle|\; \begin{array}{l} w \in W_0 \\ \mu, \nu \in \mathring{\mathfrak{h}}_{\mathbb{Z}} \\ k \in \mathbb{Z} \end{array} \right\}$$

Configuration space
$$\frac{\mathfrak{h}_{>0} - \{ x \in \mathfrak{h}_{>0} \mid \mathrm{Stab}(x) \neq \{1\} \}}{\tilde{W}}$$

[CG p. 143]

$$\begin{tikzcd} & & \tilde{\mathfrak{g}} \arrow[lld, "pr_2"'] \arrow[ld, "pr_1"'] \arrow[rd, "\nu"] & \\ \mathcal{B} & \mathfrak{g} \arrow[rd] & & \mathfrak{h} \arrow[ld] \\ & & \mathfrak{h}/W & \end{tikzcd}$$

$\tilde{\mathfrak{g}}^{sr} = \nu^{-1}(\mathfrak{h}^{reg})$

<u>Theorem</u> (van der Lek)
$$\pi_1 \left(\frac{\mathfrak{h}_{>0} - \left\{ \begin{array}{c} \text{ramification} \\ \text{divisor} \end{array} \right\}}{\tilde{W}} \right) = DAArt$$

$$\begin{array}{c} \mathfrak{h}_{>0} \\ \downarrow \\ \mathfrak{h}_{>0}/\tilde{W} \end{array}$$

is given by generators and relations (Mac 2003).

$$\left(\begin{array}{c} T_0 \ T_1 \quad\quad T_{n-1} \ T_n \\ \circ\!\!-\!\!\circ \cdots \circ\!\!-\!\!\circ \end{array} \quad \begin{array}{l} X^{\varepsilon_i} X^{\varepsilon_j} = X^{\varepsilon_j} X^{\varepsilon_i} \\ T_i X^{\varepsilon_j} = X^{\varepsilon_j} T_i \quad \text{if } j \neq i, i+1 \end{array} \right).$$

Analysis of the configuration space.

$$\widehat{W} = W_0 \ltimes \widehat{(\mathring{\mathfrak{Z}}_{\mathbb{Z}} \times \mathring{\mathfrak{Z}}_{\mathbb{Z}})} = W_0 \ltimes ((\mathbb{Z}^n \oplus \mathbb{Z}^n) \times \mathbb{Z})$$

$$\mathfrak{Z}_{>0} = \mathbb{C} \times \mathbb{C}^n \times (\mathbb{R} + i\mathbb{R}_{>0}) = \left\{ \begin{pmatrix} a \\ \lambda \\ \tau \end{pmatrix} \;\middle|\; \begin{array}{l} a \in \mathbb{C} \\ \lambda \in \mathbb{C}^n \\ \tau \in \mathbb{R} + i\mathbb{R}_{>0} \end{array} \right\}$$

$$= \mathbb{C} \times \{ \mathbb{C}^n \mid \tau \in \mathbb{R} + i\mathbb{R}_{>0} \}.$$

Then

$$\frac{\mathfrak{Z}_{>0}}{\mathring{\mathfrak{Z}}_{\mathbb{Z}} \oplus \mathring{\mathfrak{Z}}_{\mathbb{Z}}} = \mathbb{C} \times \left\{ \frac{\mathbb{C}^n}{\mathbb{Z}^n + \tau\mathbb{Z}^n} \;\middle|\; \tau \in \mathbb{R} + i\mathbb{R}_{>0} \right\}$$

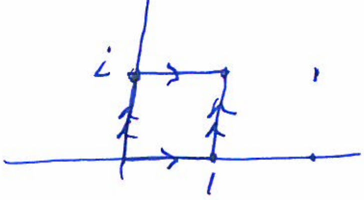
$$= \mathbb{C} \times \left\{ \left(\frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}} \right)^n \;\middle|\; \tau \in \mathbb{R} + i\mathbb{R}_{>0} \right\}$$

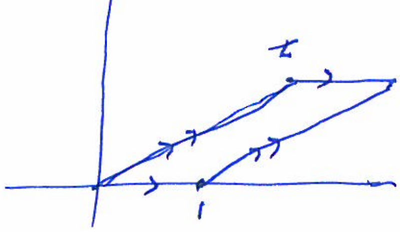
$$= \mathbb{C} \times \{ E_\tau^{\,n} \mid \tau \in \mathbb{R} + i\mathbb{R}_{>0} \}$$

Recall

$$\left\{ \begin{array}{c} \text{elliptic} \\ \text{curves} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{lattices} \\ \text{in } \mathbb{C} \end{array} \right\} / \sim \longleftrightarrow \begin{array}{l} \text{upper half} \\ \text{plane } \mathbb{R} + i\mathbb{R}_{>0} \end{array}$$

$$E_\tau = \frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}} \longleftarrow\!\shortmid \quad \mathbb{Z} + \tau\mathbb{Z} \quad \longleftarrow\!\shortmid \quad \tau$$





$E_i =$ 

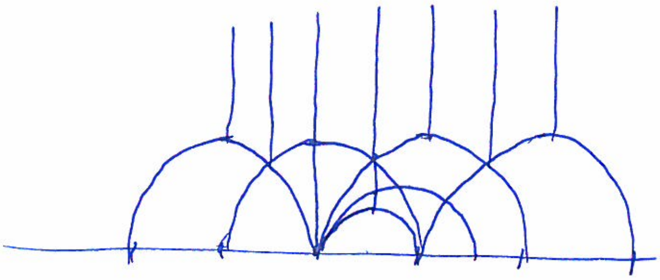
$E_\tau =$ 

$SL_2(\mathbb{Z})$ and $\mathcal{B}_3$

$SL_2(\mathbb{Z})$ acts by changing basis of the lattice $\mathbb{Z}+\tau\mathbb{Z}$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\tau = \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\tau\\1\end{pmatrix} = \begin{pmatrix}a\tau+b\\c\tau+d\end{pmatrix} = \begin{pmatrix}\frac{a\tau+b}{c\tau+d}\\1\end{pmatrix} = \frac{a\tau+b}{c\tau+b}.$$

$SL_2(\mathbb{Z})$ acts on $\mathbb{R}+i\mathbb{R}_{>0}$



The exact sequence

$$\{1\} \to \langle(\sigma_1\sigma_2\sigma_1)^4\rangle \to \mathcal{B}_3 \to SL_2(\mathbb{Z}) \to \{1\}$$

$\sigma_1 =$  $\longmapsto \begin{pmatrix}1&1\\0&1\end{pmatrix}$

$\sigma_2 =$  $\longmapsto \begin{pmatrix}1&0\\-1&1\end{pmatrix}$

makes $\mathcal{B}_3$ "the same" as $SL_2(\mathbb{Z})$.

Then

$$\frac{\mathfrak{Z}_{>0}}{\mathring{\mathfrak{Z}}_{\mathbb{Z}}\oplus\mathring{\mathfrak{Z}}_{\mathbb{Z}}} = \mathbb{C}\times\{E_\tau^n \mid \tau\in\mathbb{R}+i\mathbb{R}_{>0}\}$$

$$\frac{\mathfrak{Z}_{>0}-\{\text{stuff}\}}{\mathring{\mathfrak{Z}}_{\mathbb{Z}}\oplus\mathring{\mathfrak{Z}}_{\mathbb{Z}}} = \mathbb{C}\times\{E_\tau^n-\{\text{stuff}\} \mid \tau\in\mathbb{R}+i\mathbb{R}_{>0}\}$$

and

$$\frac{\mathfrak{Z}_{>0}-\{\text{stuff}\}}{\widetilde{W}} = \frac{\mathbb{C}}{\mathbb{Z}}\times\left\{\frac{E_\tau^n-\{\text{stuff}\}}{W_0} \,\middle|\, \tau\in\mathbb{R}+i\mathbb{R}_{>0}\right.$$

So $\mathcal{B}_3$ acts on the DAArt $\pi_1\left(\frac{\mathfrak{Z}_{>0}-\{\text{stuff}\}}{\widetilde{W}}\right)$

and the DAArt is elliptic.