Are 3-pole braids elliptic? Colloquium Dartmouth College 26/04/2014

Numbers

(1) $A, AA, AAA, AAAA, \ldots$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $AA = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $AAA = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, $AAAA = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

$\mathbb{Z}_{\geq 0}$ is the free monoid on one generator.

(2) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ make $ABBA$, $BAAA$

$$ABBA = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

$$BAAA = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$$

$SL_2(\mathbb{Z}_{\geq 0}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{matrix} a, b, c, d \in \mathbb{Z}_{\geq 0} \\ ad - bc = 1 \end{matrix} \right\}$ is the

free monoid on two generators.

(3) $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{matrix} a, b, c, d \in \mathbb{Z} \\ ad - bc = 1 \end{matrix} \right\}$ is 8 copies of $SL_2(\mathbb{Z}_{\geq 0})$:

$$SL_2(\mathbb{Z}) = SL_2(\mathbb{Z}_{\geq 0}) \sqcup s_1 SL_2(\mathbb{Z}_{\geq 0}) \sqcup SL_2(\mathbb{Z}_{\geq 0}) s_1 \sqcup s_1 SL_2(\mathbb{Z}_{\geq 0}) s_1$$
$$\sqcup (-1) SL_2(\mathbb{Z}_{\geq 0}) \sqcup (-1) s_1 SL_2(\mathbb{Z}_{\geq 0}) \sqcup (-1) SL_2(\mathbb{Z}_{\geq 0}) s_1 \sqcup (-1) s_1 SL_2(\mathbb{Z}_{\geq 0}) s_1$$

where $s_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $(-1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

(4) $GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{matrix} a, b, c, d \in \mathbb{Z} \\ ad - bc \in \mathbb{Z}^{\times} = \{\pm 1\} \end{matrix} \right\}$ is 2 copies of $SL_2(\mathbb{Z})$.

$$GL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \sqcup SL_2(\mathbb{Z}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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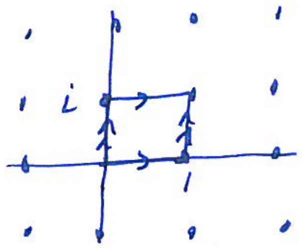
(2)

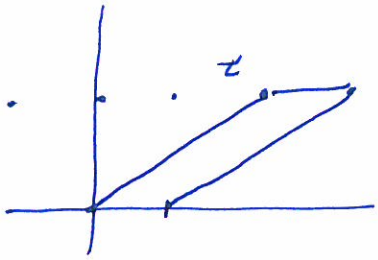
Elliptic curves

$GL_2(\mathbb{Z})$ is basis changes on $\mathbb{Z}^2 = \mathbb{Z}v_1 + \mathbb{Z}v_2$.

$$\left\{\begin{matrix}\text{Elliptic}\\ \text{curves}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{lattices}\\ \text{in } \mathbb{C}\end{matrix}\right\}/\sim \longleftrightarrow \text{upper half plane}$$

$$E_\tau = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}} \longleftarrow\!\shortmid\ \mathbb{Z}+\tau\mathbb{Z} \longleftarrow\!\shortmid\ \tau$$





$E_i =$ 

 $= E_\tau$

$SL_2(\mathbb{Z})$ acts by basis changes on $\mathbb{Z}+\tau\mathbb{Z}$, for $\tau \in \mathbb{R} + i\mathbb{R}_{>0}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau+b \\ c\tau+d \end{pmatrix} = \begin{pmatrix} \frac{a\tau+b}{c\tau+d} \\ 1 \end{pmatrix} = \frac{a\tau+b}{c\tau+d}.$$

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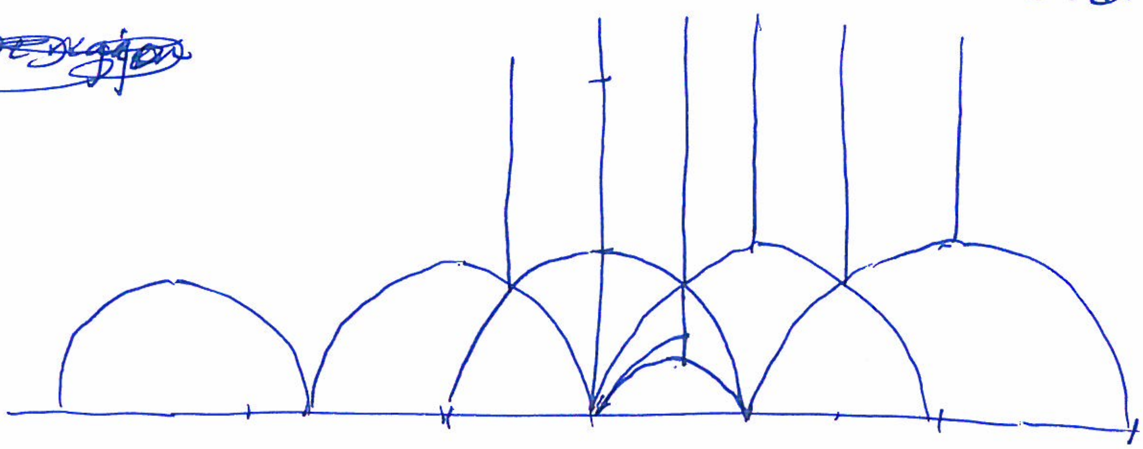
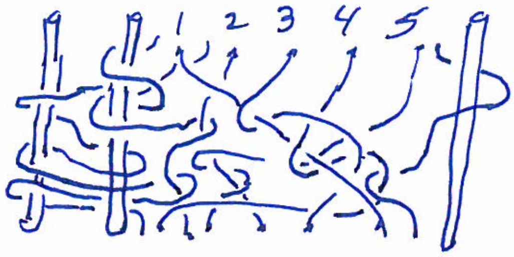


regions in the upper half plane $\longleftrightarrow$ Half of $SL_2(\mathbb{Z})$.

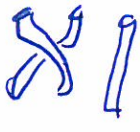
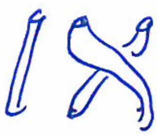
Braids with n strands and ~~one~~ ~~two~~ three poles

Hans Wenzl
Rosa Orellana
Zajj Daugherty.

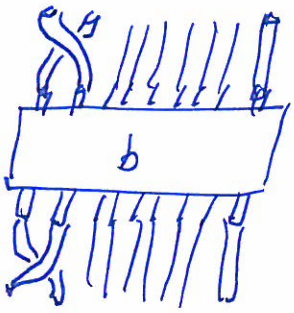
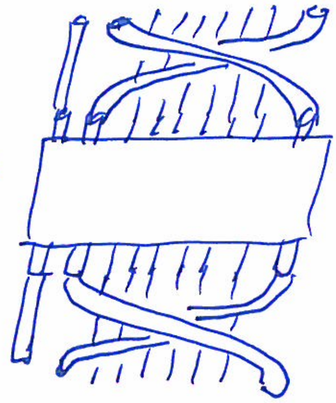


$\in \tilde{B}_3$

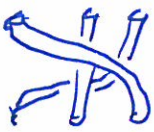
Why are 3-poles better? $\mathcal{B}_3$ has generators

$\sigma_1 =$  and $\sigma_2 =$ 

and <u>acts</u> <u>on</u> $\tilde{B}_n$:

$\sigma_1 b =$  and $\sigma_2 b =$ 

(a) $\mathcal{B}_3$ is generated by σ_1, σ_2 with relations

$\sigma_1\sigma_2\sigma_1 =$  $=$  $=$  $= \sigma_2\sigma_1\sigma_2.$

(b) $SL_2(\mathbb{Z})$ is generated by

$$\sigma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

with relations $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ and $(\sigma_1\sigma_2\sigma_1)^4 = 1$.

So $\mathcal{B}_3$ and $SL_2(\mathbb{Z})$ are <u>the same</u> (key word).

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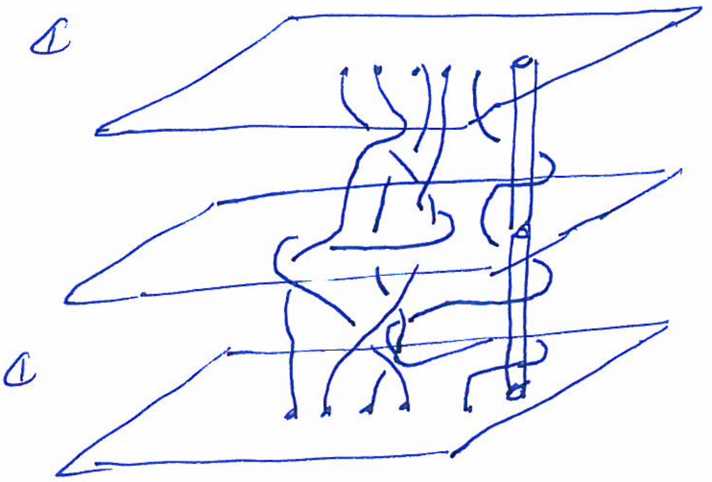

$$\sigma : [0,1] \longrightarrow \text{Bug space}$$
$$t \longmapsto (z_1(t), z_2(t), \ldots, z_n(t))$$

$$\text{Bug space} = \left\{ (z_1, \ldots, z_n) \,\middle|\, \begin{matrix} z_1, \ldots, z_n \in \mathbb{C} \\ z_i \neq z_j \end{matrix} \right\} \Big/ S_n$$

$$= \left\{ (z_1, \ldots, z_n) \,\middle|\, \begin{matrix} z_1, \ldots, z_n \in \mathbb{C} \\ z_i - z_j \neq 0 \end{matrix} \right\} \Big/ S_n$$

$$= \Big(\mathbb{C}^n - \bigcup_{i,j} H_{ij} \Big) \Big/ S_n$$

where $H_{ij} = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i - z_j = 0 \}$.

and S_n = symmetric group = permutations of the n houses.

$$\text{1-pole bug space} = \left\{ (z_1, \ldots, z_n) \,\middle|\, \begin{matrix} z_1, \ldots, z_n \in \mathbb{C} \\ z_i - z_j \neq 0, \; z_i \neq 0 \end{matrix} \right\} \Big/ S_n$$

$$= \Big(\mathbb{C}^n - \bigcup_{i,j} H_{ij} - \bigcup_i H_i \Big) \Big/ S_n .$$

where $H_i = \{ (z_1, \ldots, z_n) \mid z_1, \ldots, z_n \in \mathbb{C}^n \text{ and } z_i \neq 0 \}$.

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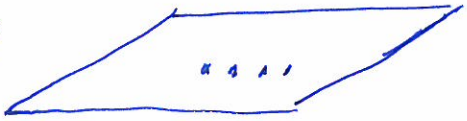
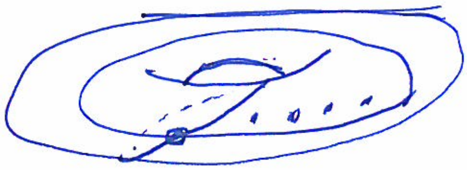
$$\begin{matrix}\text{1-pole elliptic} \\ \text{bug space}\end{matrix} = \left\{ (z_1, \ldots, z_n) \;\middle|\; \begin{matrix} z_1, \ldots, z_n \in E_\tau \\ z_i \neq z_j \text{ and } z_i \neq 0 \end{matrix} \right\} \Big/ S_n$$

$$= \left(E_\tau^n - \bigcup_{ij} H_{ij} - \bigcup_i H_i \right) \Big/ S_n$$

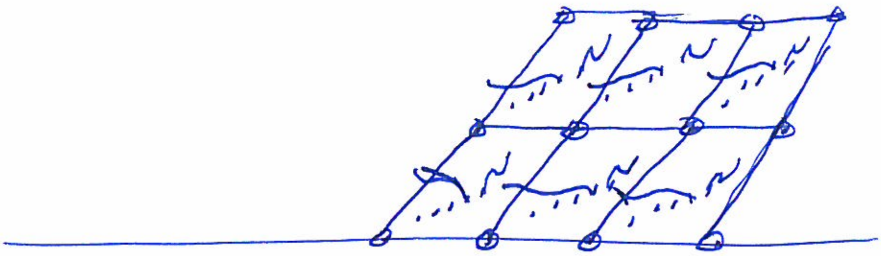
$\mathbb{C}$





E_τ

$$E_\tau = \frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}}$$



$$\begin{matrix}\text{1 pole elliptic} \\ \text{bug space}\end{matrix} = \left(\left(\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}} \right)^n - \bigcup_{ij} H_{ij} - \bigcup_i H_i \right) \Big/ S_n$$

$$= \left(\frac{\mathbb{C}^n}{(\mathbb{Z}+\tau\mathbb{Z})^n} - \{\text{stuff}\} \right) \Big/ S_n$$

$$= \left(\frac{\mathbb{C}^n - \{\text{stuff}\}}{(\mathbb{Z}+\tau\mathbb{Z})^n} \right) \Big/ S_n = \left(\mathbb{C}^n - \{\text{stuff}\} \right) \Big/ S_n \ltimes (\mathbb{Z}+\tau\mathbb{Z})^n$$

<u>Point</u>: The answer is YES.

3-Pole braids are elliptic.