The Glass Bead Game

CRM Montreal
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Categorification and Representation Theory

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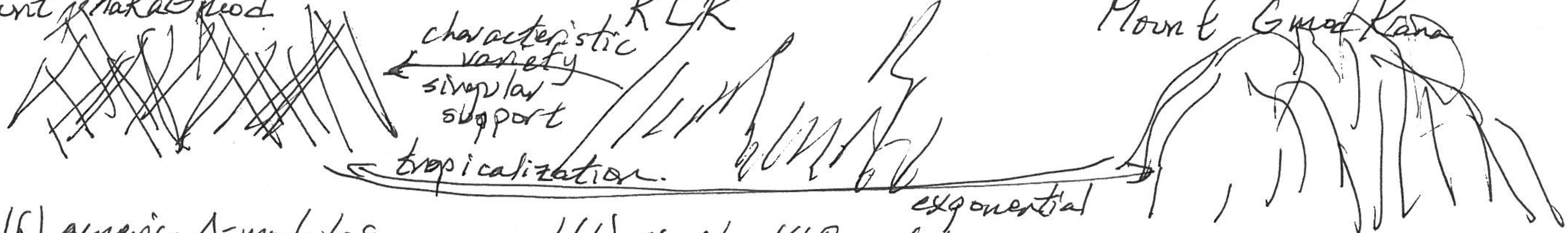


$\Lambda(b)$ generic Λ-modules

$L(b)$ simple KLR-modules

$Z(b)$ MV cycles

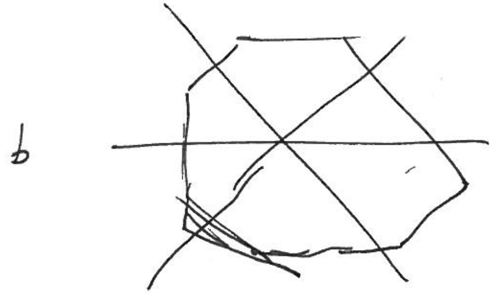
character maps

char $(\Lambda(b))$
dual semicanonical basis

char $(L(b))$
dual canonical basis

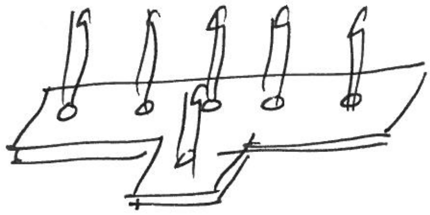
char $(Z(b))$
dual MV basis

Shadow map



MV polytopes
crystal.

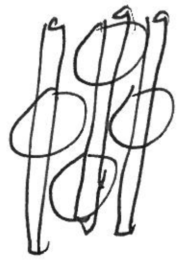
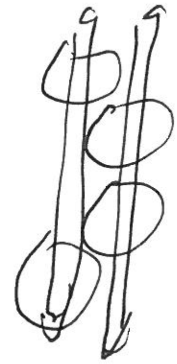
The GLS bead game



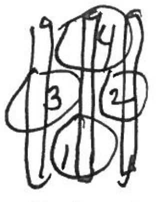
Board

Beads

A skew shape b is a configuration of beads such that any two beads on the same runner are separated by at least two beads.

if  then  or 

A standard tableau of shape b is a runner sequence $T=(i_1,i_2,\dots,i_d)$ which results in b when played.

 $= gyrg$ and  $= gryg$

r g y

Let $L(b) = \mathrm{span}\{v_T \mid T \text{ is std. tableau of shape } b\}$

with $e_u v_T = \delta_{uT} v_T$, $y_r v_T = 0$ and $\psi_j v_T = \begin{cases} v_{s_jT}, & \text{if } s_jT \text{ is std of shape } b \\ 0, & \text{otherwise} \end{cases}$

Theorem (Kleshchev-R)

$L(b)$ is a simple KLR-module.

points in Λ

$$\circ \underset{a_1^*}{\overset{a_1}{\rightleftarrows}} \circ \underset{a_2^*}{\overset{a_2}{\rightleftarrows}} \circ \underset{a_3^*}{\overset{a_3}{\rightleftarrows}} \circ \underset{a_4^*}{\overset{a_4}{\rightleftarrows}} \circ$$

Λ is generated by a_i, a_i^* with

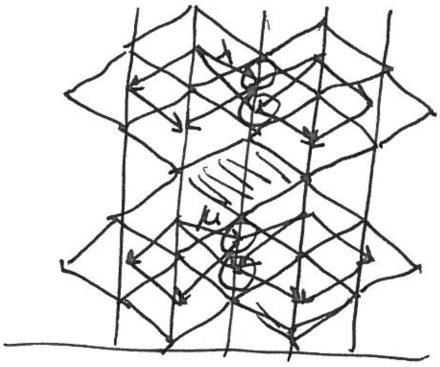
$a_1^* a_1 = 0$ 

$a_i^* a_i = a_{i+1} a_{i+1}^*$ 

$a_4^* a_4 = 0$ 

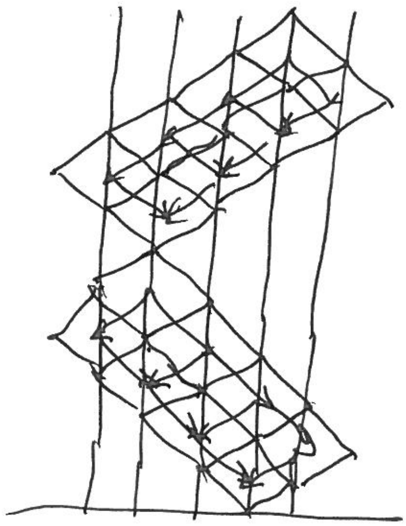
Let

$\mathbb{C}^2 \quad \mathbb{C}^4 \quad \mathbb{C}^4 \quad \mathbb{C}^4 \quad \mathbb{C}^2$

$b^{[2]} =$ 

$$\mathrm{char}(A(b^{[2]})) = \mathrm{char}(L(b^{[2]})) + \mathrm{char}(L(z))$$

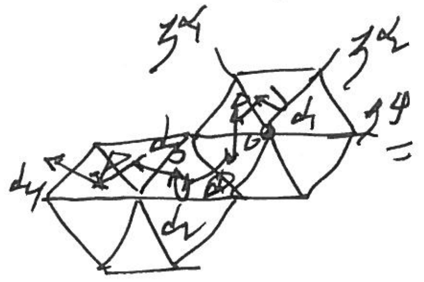
where

$z =$ 

Kashiwara-Saito, Leclerc, GLS, Williamson.

points in G/K

$G = SL_3(\mathbb{C}((t)))$ and $K = SL_3(\mathbb{C}[[t]])$.



$$= x_{-\alpha_2}(d_1 t^0)\, x_{-\alpha_1}(0)\, x_{-\varphi}(0)\, x_{-\alpha_1}(0) \cdot x_{-\varphi}(d_2 t^{-1})\, x_{-\alpha_2}(d_3 t^1)\, x_{-\alpha_1}(0) \cdot x_{-\alpha_2}(d_4 t^2)\, t_{-\alpha_1^\vee+\alpha_2^\vee} K.$$

$d_1, d_2 \in \mathbb{C}^\times$, $d_3, d_4 \in \mathbb{C}$

where

$$x_{-\alpha_1}(c) = \begin{pmatrix} 1 & & \\ c & 1 & \\ 0 & 0 & 1 \end{pmatrix} \quad x_{-\alpha_2}(c) = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & c & 1 \end{pmatrix} \quad x_{-\varphi}(c) = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ c & 0 & 1 \end{pmatrix}$$

and

$$t_{\mu_1\alpha_1^\vee+\mu_2\alpha_2^\vee} = \begin{pmatrix} t^{-\mu_1} & & \\ & t^{\mu_1-\mu_2} & \\ & & t^{\mu_2} \end{pmatrix}$$

<u>Theorem</u> (Parkinson-R-Schwer) Labeled positively folded paths index a set of coset reps. of cosets in G/K.

$$G = \bigsqcup_{\lambda^\vee \in \mathfrak{h}_\mathbb{Z}} I t_{\lambda^\vee} K \quad \text{and} \quad G = \bigsqcup_{\mu^\vee \in \mathfrak{h}_\mathbb{Z}} U t_{\mu^\vee} K$$

with $\mathfrak{h}_\mathbb{Z} = \{\mu_1\alpha_1^\vee + \mu_2\alpha_2^\vee \mid \mu_1, \mu_2 \in \mathbb{Z}\}$

$$U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{pmatrix} \,\middle|\, f_{ji} \in \mathbb{C}((t)) \right\} \quad \text{and} \quad \begin{array}{ccc} G & & \\ \cup\!\shortmid & & \\ K & \xrightarrow[\Phi]{t=0} & SL_3(\mathbb{C}) \\ \cup\!\shortmid & & \cup\!\shortmid \\ I = \Phi^{-1}(B) & \longrightarrow & B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \end{array}$$

Then

$gK \in I t_{\lambda^\vee} K \cap U t_{\mu^\vee} K$ where $\mu^\vee$ is the ending hexagon of p, $\lambda^\vee$ is the ending hexagon of the unfolding of p.

<u>The shadow map and the character map</u>

The free assoc. alg. gen. by $f_1, \ldots, f_n$ $\quad F = \bigoplus_{\gamma_i \in \mathbb{Z}_{\geq 0}} F_{-\gamma_1\alpha_1^\vee - \cdots - \gamma_n\alpha_n^\vee}$.

$h \in F_{-\gamma_1\alpha_1^\vee - \cdots - \gamma_n\alpha_n^\vee}$ then $h = \sum'_{\substack{i_1, \ldots, i_d \\ \alpha_{i_1}^\vee + \cdots + \alpha_{i_d}^\vee = -\gamma_1\alpha_1^\vee - \cdots - \gamma_n\alpha_n^\vee}} c_{i_1 \cdots i_d} f_{i_1} \cdots f_{i_d}$.

View $\begin{matrix} f_i : [0,1] \longrightarrow \mathfrak{h}_{\mathbb{R}} \\ t \longmapsto -t\alpha_i^\vee \end{matrix}$ straight line path from 0 to $-\alpha_i^\vee$.

$f_{i_1} \cdots f_{i_d}$, the concatenation, is a path 0 to $-\gamma_1\alpha_1^\vee - \cdots - \gamma_n\alpha_n^\vee$.

~~Examp~~ $\mathrm{shad}(h) = \underset{\text{hull}}{\text{Convex}} \{ f_{i_1} \cdots f_{i_d} \mid c_{i_1 \cdots i_d} \neq 0 \}$.

<u>Example</u> Type A_2 

 $= \mathrm{shad}(12 f_1 f_2 f_2 f_1 + 2 f_1 f_2 f_2 f_1)$

Let M be a Λ-module, L a KLR-module, $gK \in G/K$.

$$\left.\begin{matrix} \mathrm{char}(M) \\ \mathrm{char}(L) \\ \mathrm{char}(gK) \end{matrix}\right\} = \sum'_{i_1 \cdots i_d} c_{i_1 \cdots i_d} f_{i_1} \cdots f_{i_d} \quad \text{with}$$

$$c_{i_1 \cdots i_d} = \begin{cases} \mathrm{Card}\left\{ \begin{matrix} \text{composition series of } M \\ \text{with factors } S_{i_1}, \ldots, S_{i_d} \end{matrix} \right\} \\ \sum_{j \in \mathbb{Z}} q^j \dim(e_{i_1 \cdots i_d} L[j]), \quad \text{where } L = \bigoplus_{j \in \mathbb{Z}} L[j] \\ \mathrm{Card}\left\{ \begin{matrix} \text{expressions} \\ gK = x_{-\alpha_{i_1}}(c_1 t^{j_1}) \cdots x_{-\alpha_{i_d}}(c_d t^{j_d}) t_{\mu^\vee} K. \end{matrix} \right\} \end{cases}$$