Generalized Fock space for decomposition numbers for quantum groups at roots of unity: Conference on Representation theory and related topics (1)
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Fock space $\mathcal{P}_\ell$: Let $\ell \in \mathbb{Z}_{>0}$.

Let $\mathbb{A} = \mathbb{Z}[v, v^{-1}]$.

$$\mathcal{P}_\ell^+ = \mathbb{A}\text{-span}\{ [X_\lambda] \mid \lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^+ \}$$

with relations: If $\langle \lambda+\rho, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ then

$$[X_{s_i \circ \lambda}] = \begin{cases} -[X_\lambda], & \text{if } \langle \lambda+\rho, \alpha_i^\vee \rangle \in \ell\mathbb{Z}_{\geq 0}, \\ -v[X_\lambda], & \text{if } 0 < \langle \lambda+\rho, \alpha_i^\vee \rangle < \ell, \\ -v[X_{s_i \circ \lambda^{(1)}}] - [X_{\lambda^{(1)}}] - v[X_\lambda], & \text{otherwise} \end{cases}$$

where $\lambda^{(1)} = \lambda - j\alpha_i$ with $\langle \lambda+\rho, \alpha_i^\vee \rangle = k\ell + j$ with $0 \leq j < \ell$.

and bar involution $\bar{\ }: \mathcal{P}_\ell^+ \to \mathcal{P}_\ell^+$

$$\bar{v} = v^{-1} \text{ and } \overline{[X_\lambda]} = v^{\ell(w_0)}(-v^{-1})^{\ell(w_0)}[X_{w_0 \circ \lambda}]$$

where $\ell(w_0) = \text{Card}\{ \text{pos roots } \alpha \mid \langle \lambda+\rho, \alpha_i^\vee \rangle \in \ell\mathbb{Z}_{\geq 0} \}$

w_0 = longest element of W_0.

$\mathcal{P}_\ell^+$ has bases

$$\{[X_\lambda] \mid \lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^+\} \text{ and } \{[C_\lambda] \mid \lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^+\}$$

where $\overline{[C_\lambda]} = [C_\lambda]$ and

$$[C_\lambda] = [X_\lambda] + \sum_{\substack{\mu < \lambda \\ \mu \in (\mathfrak{h}_{\mathbb{Z}}^*)^+}} Q_{\lambda\mu}[X_\mu], \text{ with } Q_{\lambda\mu} \in v\mathbb{Z}[v].$$

"Graded Grothendieck groups"

$$K_v(U_\varepsilon \mathfrak{g}_0\text{-fd mod}) \longrightarrow \mathcal{P}_\ell^+$$

Weyl module $\quad \Delta(\lambda) \longmapsto [X_\lambda]$

simple module $\quad L(\lambda) \longmapsto [C_\lambda]$

and

$$K_v(\mathcal{O}_{\mathfrak{g}_0}^{\mathfrak{g}}[-\ell-h]) \longrightarrow \mathcal{P}_\ell^+$$

Weyl module $\quad \Delta(\lambda) \longmapsto [X_\lambda]$

simple module $\quad L(\lambda) \longmapsto [C_\lambda]$

$U_\varepsilon\mathfrak{g}_0$-fd mod = category of fin. dim quantum group module at $\varepsilon^\ell = 1$,

$\mathcal{O}_{\mathfrak{g}_0}^{\mathfrak{g}}[-\ell-h]$ = parabolic category $\mathcal{O}$ for $\mathfrak{g} = \mathfrak{g}_0[z, z^{-1}] \oplus \mathbb{C}K$ at level $-\ell - \rho(K)$ where $\langle \rho, \alpha_i^\vee \rangle = 1$ for $i = 0, 1, \dots, n$.

$K(U_\varepsilon\mathfrak{g}_0\text{-fd mod})$ = Grothendieck group of $U_\varepsilon\mathfrak{g}_0$-mod.

"graded" = K_v means

$$[X_\lambda] = \sum_{\mu \le \lambda} Q_{\lambda\mu} [C_\mu] \quad \text{with}$$

$$Q_{\lambda\mu} = \sum_{j \in \mathbb{Z}_{\geq 0}} [\Delta^{(j)}/\Delta^{(j+1)} : L(\mu)]\, v^j \quad \text{where}$$

$\Delta(\lambda) \supseteq \Delta(\lambda)^{(1)} \supseteq \Delta(\lambda)^{(2)} \supseteq \cdots$ is a Jantzen filtration with respect to the Shapovalov form on $\Delta(\lambda)$.

Modules for Translation functor algebras

Classical examples:

(1) $\mathfrak{g}$ = fin. dim. complex reductive Lie algebra

$\mathcal{O}[\nu]$ = block of category $\mathcal{O}$ containing $[L(\nu)]$.

$$K(\mathcal{O}[\nu]) = \operatorname{span}\{[L(w\circ\nu)] \mid w \in W_0\}$$

Wall crossing functors (special translation functors)

$F_{s_1}, \dots, F_{s_n}$ provide

an action of the Hecke algebra H_v of W_0 on

$\operatorname{span}\{T_w \mid w \in W_0\}$, where $T_w = [M(w\circ\nu)]$

where $M(\mu)$ denotes a Verma module.

$$F_{s_i} T_w = [F_{s_i} M(w\circ\nu)]$$

$$= \begin{cases} [M(s_i w\circ\nu)], & \text{if } \ell(s_i w) > \ell(w), \\ [M(s_i w\circ\nu)] + v[M(w\circ\nu)], & \text{if } \ell(s_i w) < \ell(w). \end{cases}$$

$$= \begin{cases} T_{s_i w}, & \text{if } s_i w > w, \\ T_{s_i w} + v T_w, & \text{if } s_i w < w. \end{cases}$$

(2) $\mathfrak{g} = \mathfrak{gl}_n$ and $\varepsilon^\ell = 1$.

$$K(U_\varepsilon\mathfrak{g}\text{-fdmod}) = \mathrm{span}\{[L(\lambda)] \mid \lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^+\}$$

Translation functors coming from tensoring by $\mathbb{C}^n$

F_j, $j = 0, 1, 2, \dots, \ell-1$, provide

an action of $U_v \widehat{\mathfrak{sl}}_\ell$ on

$\mathrm{span}\{|\lambda\rangle \mid \lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^+\}$ where $|\lambda\rangle = [\Delta(\lambda)]$

with $\Delta(\lambda)$ denoting a Weyl module.

$$F_j|\lambda\rangle = \sum_\mu v^{MM(\mu,\lambda,j)}|\mu\rangle$$

"Crystals"

The crystal is $\{[L(w \circ v)] \mid w \in W_0\}$

$\{[L(\lambda)] \mid \lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^+\}$

with operators

$$\tilde{F}_{s_i}[L(w \circ v)] = [\mathrm{soc}(F_{s_i}(L(w \circ v)))]$$

$$\tilde{F}_j[L(\lambda)] = [\mathrm{soc}(F_j(L(\lambda)))].$$

<u>TODO</u>

(1) $\mathfrak{g}_0 = \mathfrak{sp}_n$ or $\mathfrak{so}_n$ and $\epsilon^\ell = 1$.

$$K(U_\epsilon\mathfrak{g}_0\text{-fdmod}) = \text{span}\{[L(\lambda)] \mid \lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^+\}$$

Translation functors coming from tensoring by $\mathbb{C}^n$

F_j, $j = ????$ provide

an action of ???? on

$\text{span}\{|\lambda\rangle \mid \lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^+\}$ where $|\lambda\rangle = [\Delta(\lambda)]$

with $\Delta(\lambda)$ denoting a Weyl module.

$$F_j|\lambda\rangle = \sum_\mu v^{???} |\mu\rangle.$$

(2) Generalize Etingof-Kirillov

from $\mathfrak{g}_0 = \mathfrak{gl}_n$ to $\mathfrak{g}_0$ = fin. dim. complex reductive

to show that 2 parameters v and ℓ, and

$$P_\ell^+ = \mathbb{Z}[v, v^{-1}]\text{-span}\{|\lambda\rangle \mid \lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^+\}$$

work together to produce Macdonald polynomials.