Affine and degenerate affine BMW algebras

Workshop on diagram algebras Sept. 8-12, 2014.

enveloping algebra	$U\mathfrak{g}$	quantization $\longrightarrow$	$U_h\mathfrak{g}$	quantum group
centre	$Z(U\mathfrak{g})$	$\longleftarrow$ degeneration.	$Z(U_h\mathfrak{g})$	centre
deg. affine braid algebra	$\mathcal{B}_k$		B_k	affine braid group
deg. affine BMW	$\mathcal{W}_k$		W_k	affine BMW

Parameters $q = e^{h/2}$, $z = \epsilon q^{y}$

$$\epsilon = \begin{cases} +1, & \text{if } \mathfrak{g} = \mathfrak{so}_n \\ -1, & \text{if } \mathfrak{g} = \mathfrak{sp}_n \end{cases} \qquad y = \begin{cases} 2r, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \\ 2r+1, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r} \\ 2r-1, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r} \end{cases}$$

Base rings $\mathbb{K} = Z(U\mathfrak{g})$ and $\mathbb{C} = Z(U_h\mathfrak{g})$.

$$\mathbb{K} = \{ f \in \mathbb{C}[h_1, h_2, \dots, h_r]^{S_r} \mid f(h_1, h_2, \dots, h_r) = f(-h_1, h_2, \dots, h_r) \}$$

$$C = \{ f \in \mathbb{C}[L_1^{\pm 1}, \dots, L_r^{\pm 1}]^{S_r} \mid f(L_1, L_2, \dots, L_r) = f(L_1^{-1}, L_2, \dots, L_r) \}$$

R-matrix $R \in U_h\mathfrak{g} \otimes U_h\mathfrak{g}$ and $\gamma = \sum_b b \otimes b^* \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$

Higher Casimirs Fix a fin. dim'l $\mathfrak{g}$-module V.

$$z_0^{(\ell)} = \epsilon(id \otimes tr_V)\left(\left(\tfrac{1}{2}y + \gamma\right)^{\ell}\right), \quad \text{for } \ell \in \mathbb{Z}_{\geq 0}$$

$$Z_0^{(\ell)} = \epsilon(id \otimes qtr_V)\left((zR_{21}R)^{\ell}\right), \quad \text{for } \ell \in \mathbb{Z}.$$

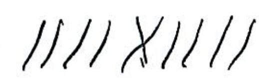
Admissibility Let $z_0(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} z_0^{(\ell)} u^{-\ell}$. Then

$$\left(z_0(+u) + \epsilon u - \tfrac{1}{2}\right)\left(z_0(-u) - \epsilon u - \tfrac{1}{2}\right) = \left(\epsilon u - \tfrac{1}{2}\right)\left(-\epsilon u - \tfrac{1}{2}\right)$$

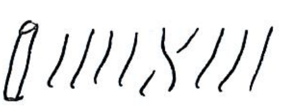
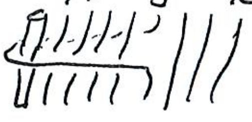
Perelomov-Popov

$$\left(z_0(u) + \epsilon u - \tfrac{1}{2}\right) = \left(\epsilon u + \tfrac{1}{2}\right)\left(\frac{u + \frac{1}{2}y - r}{u + \frac{1}{2}y + r}\right) \prod_{i=1}^{r} \frac{(u + h_i + \frac{1}{2})(u - h_i + \frac{1}{2})}{(u + h_i - \frac{1}{2})(u - h_i - \frac{1}{2})}$$

The deg. affine braid algebra $\mathcal{B}_k$ has generators

$t_{s_i} =$  and $y_j =$  and κ_0, κ_1

The group algebra of the affine braid group $\mathcal{B}_k$ has generators

$T_{s_i} =$ , $Y_j =$ 

and Y_j^{-1}.

Define $e_i \in \mathcal{B}_k$ and $E_i \in \mathcal{B}_k$ by

$t_{s_i} y_i = y_{i+1} t_{s_i} - (1-e_i)$ and $T_{s_i} Y_i = Y_{i+1} T_{s_i} - (q-q^{-1}) Y_{i+1}(1-E_i)$

The deg. affine BMW algebra $\mathcal{W}_k$ is $\mathcal{B}_k$ with

$t_{s_i} e_i = e_i t_{s_i} = e_i$ and $e_i t_{s_{i-1}} e_i = e_i t_{s_{i+1}} e_i = \epsilon e_i$.

$e_1 y_1^{\ell} e_1 = z_0^{(\ell)} e_1$ and $e_i(y_i + y_{i+1}) = 0 = (y_i + y_{i+1}) e_i$.

The affine BMW algebra W_k is B_k with

$T_{s_i}^{\pm 1} E_i = E_i T_{s_i}^{\pm 1} = z^{\mp 1} E_i$, $E_i T_{s_{i-1}}^{\pm 1} E_i = E_i T_{s_{i+1}}^{\pm 1} E_i = z^{\pm 1} E_i$

$E_1 Y_1^{\ell} E_1 = z_0^{(\ell)} E_1$, $E_i Y_i Y_{i+1} = Y_i Y_{i+1} E_i = E_i$.

Action on tensor space

$\mathfrak{g}$ complex fin dim'l reductive Lie algebra

M and V are $\mathfrak{g}$-modules.

$\mathfrak{g} = \mathfrak{sp}_n$ or $\mathfrak{so}_n$ or $\mathfrak{sl}_n$ or $\mathfrak{gl}_n$ and $V = \mathbb{C}^n = L(\omega_1)$.

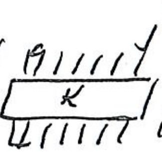
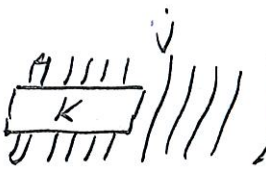
There are algebra homomorphisms

$$\mathcal{B}_k \longrightarrow \mathrm{End}_{U\mathfrak{g}}(M \otimes V^{\otimes k}), \qquad \mathcal{B}_k \longrightarrow \mathrm{End}_{U_h\mathfrak{g}}(M \otimes V^{\otimes k})$$

$$\mathcal{W}_k \longrightarrow \mathrm{End}_{U\mathfrak{g}}(M \otimes V^{\otimes k}), \qquad W_k \longrightarrow \mathrm{End}_{U_h\mathfrak{g}}(M \otimes V^{\otimes k})$$

given by

$\kappa_0 =$ , $\kappa_1 =$  with $\kappa = \sum_b b b^*$.

$y_j = \frac{1}{2}\Big($  $-$  $\Big)$

and $\check{R}_{MN} : M \otimes N \longrightarrow N \otimes M$ is 

$$\begin{array}{cccc}
\text{affine BMW} & W_k & \longrightarrow\!\!\!\!\rightarrow & \dfrac{W_k}{\langle (Y_1 - b_1)\cdots(Y_1 - b_m)\rangle} & \text{cyclotomic BMW} \\
& \downarrow & & \downarrow & \\
\text{affine Hecke} & \dfrac{W_k}{\langle E_1 \rangle} & \longrightarrow\!\!\!\!\rightarrow & \dfrac{W_k}{\langle E_1, (Y_1 - b_1)\cdots(Y_1 - b_m)\rangle} & \text{cyclotomic Hecke.}
\end{array}$$

Bases $\hat{B}_k = \{\text{Brauer diagrams on } k\text{-dots}\}$

$\mathcal{W}_k$ has $\mathbb{K}$ basis $\{d(n_1,\ldots,n_k) \mid d \in \hat{B}_k,\ n_1,n_2,\ldots n_k \in \mathbb{Z}_{\geq 0}\}$

W_k has $\mathbb{C}$ basis $\{\underline{d}(n_1,\ldots,n_k) \mid d \in \hat{B}_k,\ n_1,n_2,\ldots,n_k \in \mathbb{Z}_{\geq 0}\}$

$d =$  $\underline{d} =$ 

$d(n_1,n_2,n_3,n_4,n_5) = y_1^{n_1} y_2^{n_2} y_3^{n_3} y_5^{n_4} d\, y_2^{n_5}$ Nazarov

Ariki-Mathas-Rui.

$\underline{d}(n_1,n_2,n_3,n_4,n_5) = Y_1^{n_1} Y_2^{n_2} Y_3^{n_3} Y_5^{n_4} \underline{d}\, Y_2^{n_5}$. Goodman-Mosley

Center

$$Z(\mathcal{W}_k) = \left\{ f \in \mathbb{K}[y_1,y_2,\ldots,y_k]^{S_k} \;\middle|\; \begin{array}{l} f(y_1,-y_1,y_3,\ldots,y_k) \\ = f(0,0,y_3,\ldots,y_k) \end{array} \right\}$$

$$= \mathbb{K}[p_1,p_3,p_5,\ldots] \text{ with } p_i = y_1^i + y_2^i + \cdots + y_k^i.$$

$$Z(W_k) = \left\{ f \in \mathbb{C}[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}]^{S_k} \;\middle|\; \begin{array}{l} f(Y_1,Y_1^{-1},Y_3,\ldots,Y_k) \\ = f(1,1,Y_3,\ldots,Y_k) \end{array} \right\}$$

$$= \mathbb{C}[e_k^{\pm 1}][p_1^-,p_2^-,\ldots] \text{ with } e_k = Y_1 Y_2 \cdots Y_k$$

and $p_i^- = Y_1^i + Y_2^i + \cdots + Y_k^i - (Y_1^{-i} + Y_2^{-i} + \cdots + Y_k^{-i})$.

Please Provide thoROugh Beautiful Lucid Explanation or ManuScript

(1) Explain why $Z(W_k) = H^*_T$(isotropic Grassmannian)

and $Z(W_k) = K_T$(isotropic Grassmannian) Ikeda-Naruse

(2) Generalize from $\mathfrak{g} = so_n$ and sp_n to the p-compact groups with Weyl groups $G(m,1,n)$.

(3) Explain why the formulas for $z_0 (u)$ (and others) are similar to Yangian formulas and provide Yangian $\longrightarrow$ def. affine tantalizer transfer.

(4) Show that "Irreducible affine BMW representations are indexed by aperiodic multisegments with $k, k-2, k-4, \ldots$ boxes".

(5) Rework everything for graded/KLR BMW and provide "Brundan-Kleshchev" isomorphisms.

(6) Compute the Jantzen determinants for translation functors coming from $\cdot \otimes V$ and provide an "LLT algorithm" for graded decomposition numbers.