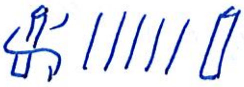
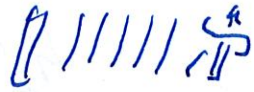
Two Boundary Hecke Algebras and Schur-Weyl duality ①

Two pole braids ANU Kioloa 12.11.2014

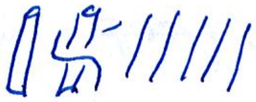
Generators: $L =$  , $R =$ 

$T_i =$  for $i \in \{1, \dots, k-1\}$

Relations: $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

$L T_1 L T_1 = T_1 L T_1 L$, $\quad R T_{k-1} R T_{k-1} = T_{k-1} R T_{k-1} R$.

Rearrange the poles

$X_1 =$  , $Y_1 =$  , $T_i =$ 

and add $P =$  to get B_k^{ext}.

<u>The Hecke algebra</u> $H_k^{ext} = \mathbb{C}B_k^{ext}$ with

$$(X_1 - a_1)(X_1 - a_2) = 0, \quad (Y_1 - b_1)(Y_1 - b_2) = 0$$

$$(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0.$$

<u>Remark</u> Representations of H_k^{ext}

$$\frac{K(\text{gen Springer fiber})}{\text{max. proper submodule}} = \text{simple } H_k^{ext} \text{ modules}$$

where $G = Sp_{2k}(\mathbb{C})$ or use exotic nil cone.

(Kazhdan-Lusztig, Kato).

Another presentation of H_k^{ext}

The finite Hecke algebra of type C is

Generators: $T_0, T_1, \dots, T_{k-1}$ with

$$T_0T_1T_0T_1 = T_1T_0T_1T_0, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$$

$$(T_0 - t_0^{-\frac{1}{2}})(T_0 + t_0^{\frac{1}{2}}) = 0, \quad (T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0.$$

__Theorem__ $H_k^{ext} = \mathbb{C}[W_0^{\pm 1}, \dots, W_k^{\pm 1}] \otimes H_0.$

with

$$T_0 W^\lambda = W^{s_0\lambda}T_0 + \left((t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-1}\right)\frac{W^\lambda - W^{s_0\lambda}}{1 - W_1^{-2}}$$

$$T_i W^\lambda = W^{s_i\lambda}T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\frac{W^\lambda - W^{s_i\lambda}}{1 - W_iW_{i+1}^{-1}}.$$

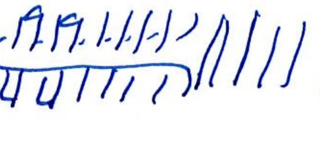
where $W^\lambda = W_0^{\lambda_0}W_1^{\lambda_1}\cdots W_k^{\lambda_k}$ for $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_k) \in \mathbb{Z}^{k+1}$

$s_0\lambda = (\lambda_0, -\lambda_1, \lambda_2, \dots, \lambda_k)$ and

$s_i\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \dots, \lambda_k)$

and $t_0^{\frac{1}{2}} = a_1^{+\frac{1}{2}}(-a_2)^{-\frac{1}{2}}, \quad t_k^{\frac{1}{2}} = b_1^{+\frac{1}{2}}(-b_2)^{-\frac{1}{2}}.$

$W_0 = PW_1W_2\cdots W_k$ and $W_i = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}b_1^{-\frac{1}{2}}(-b_2)^{-\frac{1}{2}}z_i$

with $z_i =$ , $\quad T_0 = b_1^{-\frac{1}{2}}(-b_2)^{-\frac{1}{2}}y_1$

Action of H_k^{ext} on $M\otimes N\otimes V^{\otimes k}$

$\mathfrak{g}=\mathfrak{gl}_n$ and fix $U_q\mathfrak{gl}_n$-modules

$$M = L\left(\boxed{}\right) = L(a^c) \qquad N = L\left(\boxed{}\right) = L(b^d)$$

$$V = L(\square).$$

R-matrices

$\check{R}_{VV} : V\otimes V \to V\otimes V$, $\check{R}_{VV} =$

$\in \mathrm{End}_{U_q\mathfrak{g}}(V\otimes V)$

and $\check{R}^2_{MV} =$

, $\check{R}^2_{NV} =$

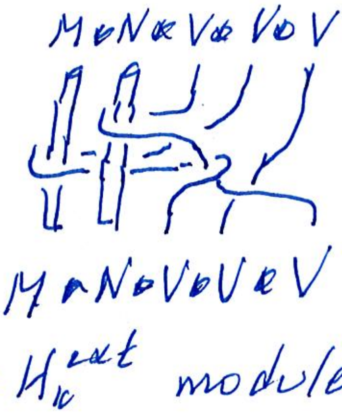
, $\check{R}^2_{MN} =$

.

Let

$$a_1 = q^{2a},\quad a_2 = q^{-2c},\quad b_1 = q^{2b},\quad b_2 = q^{-2d}.$$

Then H_k^{ext} acts on $M\otimes N\otimes V^{\otimes k}$:



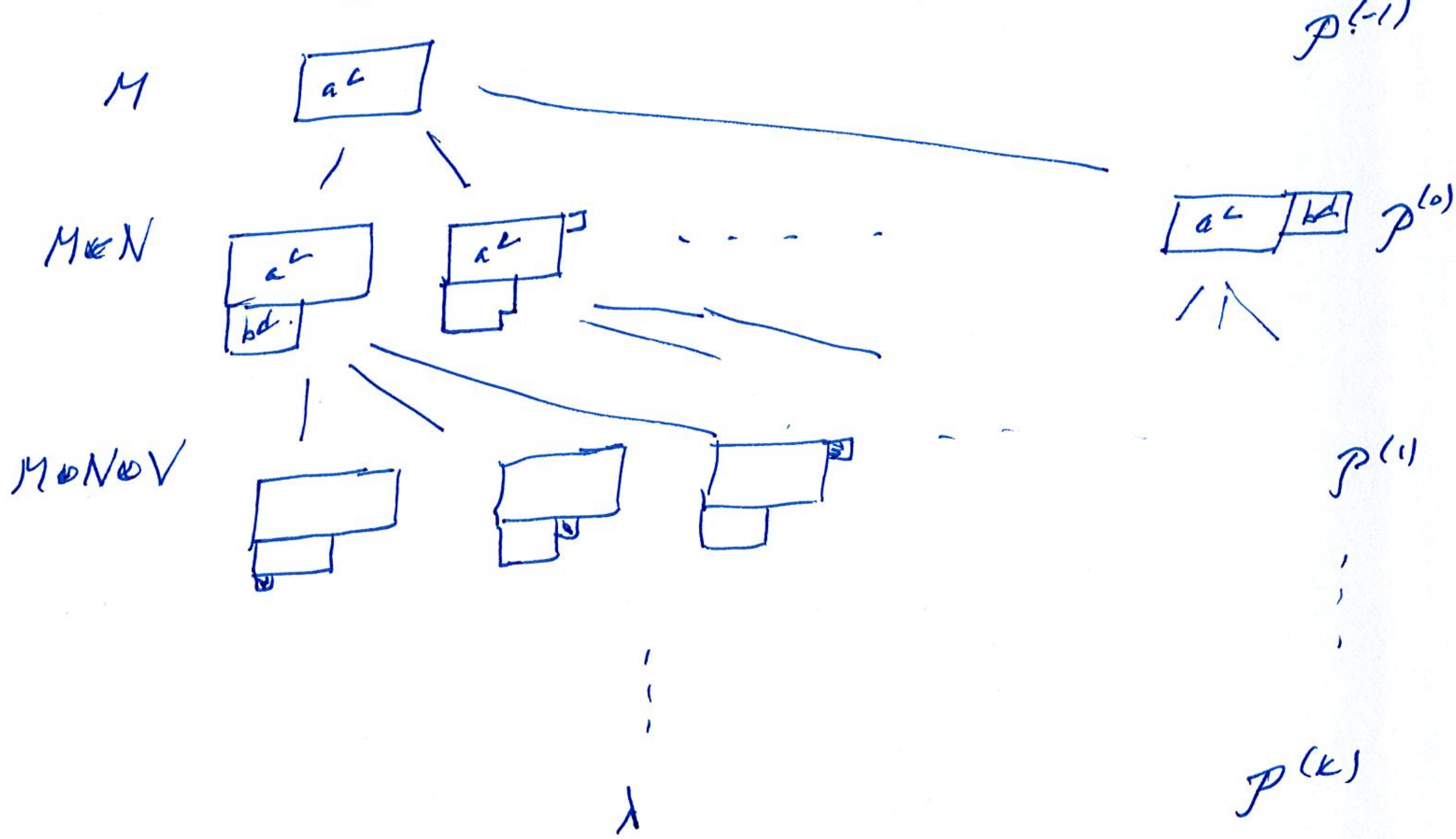
As $U_q\mathfrak{g}\times H_k^{ext}$ modules

$$M\otimes N\otimes V^{\otimes k} = \bigoplus_{\lambda\in\mathcal{P}^k} L(\lambda)\otimes H_k^\lambda$$

simple $U_q\mathfrak{g}$ module.

simple H_k^λ-module

Bratelli diagram $\mathcal{P}$

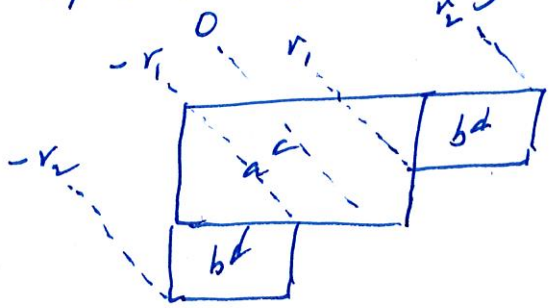


$\dim(H_k^\lambda) = \#$ paths $\boxed{a^c} \subseteq S^{(0)} \subseteq S^{(1)} \subseteq \cdots \subseteq S^{k} = \lambda$ in the Bratelli diagram.

The shifted content of a box is

$$\tilde{c}(\text{box}) = (\text{diagonal no.}) - \tfrac{1}{2}(a-c+b-d)$$

Let r_1 and r_2 be given by



shifted contents

Characters of H_k^λ

$$H_k^\lambda = \bigoplus_\gamma (H_k^\lambda)_\gamma^{gen} \quad \text{with}$$

$(H_k^\lambda)_\gamma^{gen} = \{ m \in M \mid (W_i - \gamma_i)^\ell m = 0 \text{ for some } \ell \in \mathbb{Z}_{>0} \}$.

and $\quad \mathrm{char}(H_k^\lambda) = \sum_\gamma \dim((H_k^\lambda)_\gamma^{gen}) \, \mathrm{e}^\gamma \quad$ (generating function).

The positive roots for type C_k are

$$R^+ = \{ \varepsilon_j \mid j \in \{1, \dots k\} \} \cup \{ \varepsilon_j - \varepsilon_i \mid i < j \} \cup \{ \varepsilon_j + \varepsilon_i \mid i < j \}$$

and if w is a signed permutation of $1, 2, \dots k$ then

$$R(w) = \{ \alpha \in R^+ \mid w\alpha \notin R^+ \} \quad \text{where } \varepsilon_{-i} = -\varepsilon_i .$$

Given $\lambda \in \mathbb{R}^{(k)}$ and a, b, c, d we give a construction of

$$\underline{c} = (c_0, c_1, \dots, c_k) \quad \text{and} \quad J \subseteq P(\underline{c})$$

with $0 \le c_1 \le c_2 \le \cdots \le c_k$, $\underline{c} \in \mathbb{Z}^{k+1}$ and

$$P(\underline{c}) = \{ \varepsilon_j \mid c_j \in \{r_1, r_2\} \} \cup \{ \varepsilon_j - \varepsilon_i \mid c_j = c_i + 1 \}$$
$$\cup \{ \varepsilon_j + \varepsilon_i \mid c_j = -c_i + 1 \}.$$

<u>Theorem</u> $\quad \dim((H_k^\lambda)_\gamma^{gen}) = \begin{cases} 1, & \text{if } \gamma \in \mathcal{F}^{(\underline{c}, J)} \\ 0, & \text{if } \gamma \notin \mathcal{F}^{(\underline{c}, J)} \end{cases}$

where

$$\mathcal{F}^{(\underline{c}, J)} = \left\{ \gamma = (q^{c_0}, q^{c_{w^{-1}(1)}}, \dots, q^{c_{w^{-1}(k)}}) \;\middle|\; \begin{array}{l} w \text{ a signed perm with} \\ R(w) \cap P(\underline{c}) = J \end{array} \right\}$$

with $c_{-i} = -c_i$

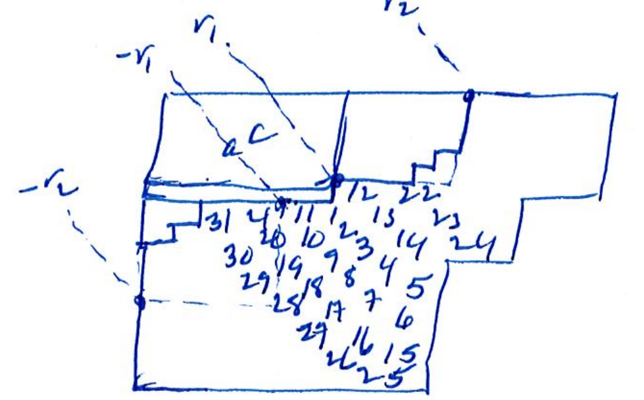
Let

$$c_0 = \sum_{\mathrm{box} \in \lambda} c(\mathrm{box}) - \frac{k}{2}(a-c+b-d) - \frac{ac}{2}(a-c) - \frac{bd}{2}(b-d)$$

and let

$$S = (\boxed{a^c} \subseteq S^{(0)} \subseteq S^{(1)} \subseteq \cdots \subseteq S^{(k)} = \lambda)$$

be a path to $\lambda \in P^{(k)}$



numbering of boxes.

Let $J = \{ \varepsilon_j \mid \tilde{c}(\mathrm{box}_j) \in \{-r_1, -r_2\} \}$

$$\sqcup \left\{ \varepsilon_j - \varepsilon_i \;\middle|\; i<j \text{ and } \begin{array}{l} \tilde{c}(\mathrm{box}_j) = \tilde{c}(\mathrm{box}_i) + 1 > 0 \text{ and } \mathrm{box}_j \text{ is NW of } \mathrm{box}_i \\ \underline{\text{or}}\ \tilde{c}(\mathrm{box}_j) = \tilde{c}(\mathrm{box}_i) - 1 < 0 \text{ and } \mathrm{box}_j \text{ is SE of } \mathrm{box}_i \\ \underline{\text{or}}\ \tilde{c}(\mathrm{box}_j) = -\tilde{c}(\mathrm{box}_i) - 1 < 0 < \tilde{c}(\mathrm{box}_i) \end{array} \right\}$$

$$\sqcup \left\{ \varepsilon_j + \varepsilon_i \;\middle|\; i<j \text{ and } \begin{array}{l} \tilde{c}(\mathrm{box}_j) = 1, \tilde{c}(\mathrm{box}_i) = 0 \\ \tilde{c}(\mathrm{box}_j) = -1, \tilde{c}(\mathrm{box}_i) = 0 \\ \tilde{c}(\mathrm{box}_j) = \tfrac{1}{2}, \tilde{c}(\mathrm{box}_i) = \tfrac{1}{2} \\ \tilde{c}(\mathrm{box}_j) = -\tfrac{1}{2}, \tilde{c}(\mathrm{box}_i) = \tfrac{1}{2} \\ \tilde{c}(\mathrm{box}_j) = \tfrac{1}{2}, \tilde{c}(\mathrm{box}_i) = -\tfrac{1}{2} \\ \tilde{c}(\mathrm{box}_j) = -\tfrac{1}{2}, \tilde{c}(\mathrm{box}_i) = -\tfrac{1}{2} \end{array} \right.$$

$w(i) > 0$, $w(j) > 0$ so $-w(i) \not> w(j)$ ↯.

$w(i) > 0$, $w(j) < 0$ — $w(i) < -w(j)$, $w(j) < -w(i)$; $w(i) > -w(j)$, $w(j) > -w(i)$

$0 < w(i) < w(j)$ so $-w(i) \not> w(j)$ ↯.

↯ $w(j) < -w(i)$; $-w(i) < w(j)$

$w(i) < 0$, $w(j) > 0$

$0 > w(j)$, $0 > w(i)$ $-w(j) < -w(i) < 0$ so $-w(i) \not> w(j)$ ever.

$$R(w) = \{ \varepsilon_j \mid w(i) < 0 \} \sqcup \{ \varepsilon_j - \varepsilon_i \mid w(i) > w(j) \}$$

$$\sqcup \{ \varepsilon_j + \varepsilon_i \mid -w(i) > w(j) \}$$

$$P(\leq) = \{ \varepsilon_j \mid c_j \in \{r_1, r_2\} \} \sqcup \{ \varepsilon_j - \varepsilon_i \mid c_j = c_i + 1 \} \sqcup \{ \varepsilon_j + \varepsilon_i \mid c_j = -c_i + 1 \}$$