Moduli spaces seminar 28.11.2014 Univ. of Melbourne ⓪

The Peterson isomorphism from Lam–Shimozono

$H_T^t(Gr_G) = H_T(Gr_G)[\xi_t^{-1} \mid t \in \tilde{Q}]$ where

Gr_G is the loop Grassmannian.

$QH_q^T(G/B) = QH^T(G/B)[q_i^{-1} \mid i \in I]$

Theorem $H_T^t(Gr_G) \longrightarrow QH_q^T(G/B)$ is an S-algebra isomorphism.

$\xi_{wt_\lambda} \xi_{t_\mu}^{-1} \longmapsto q_{\lambda-\mu} \sigma^w$

Theorem 3 Lecture 12 = Theorem 3 Lecture 11: Let

$$x_1 = w_1 t_{h_1} \in W_{af}^+ \quad \text{and} \quad x_2 = w_2 t_{h_2} \in W_{af}^+$$

Then we have mutually inverse isomorphisms

$$B_{af}^- x_1 B_{af} \cap B_{af} x_2 B_{af} \underset{\pi_+}{\overset{\pi_-}{\rightleftarrows}} M^{w_1, w_2}_{G/B,\, \pi_B(h_1 - h_2)}$$

given by

$\pi_-(g B_{af}) = \pi_B(g)$, if $g \in B_{af}^- x_1$ such that $g B_{af} \in B_{af}^- x_1 B_{af} \cap B_{af} x_2 B_{af}$

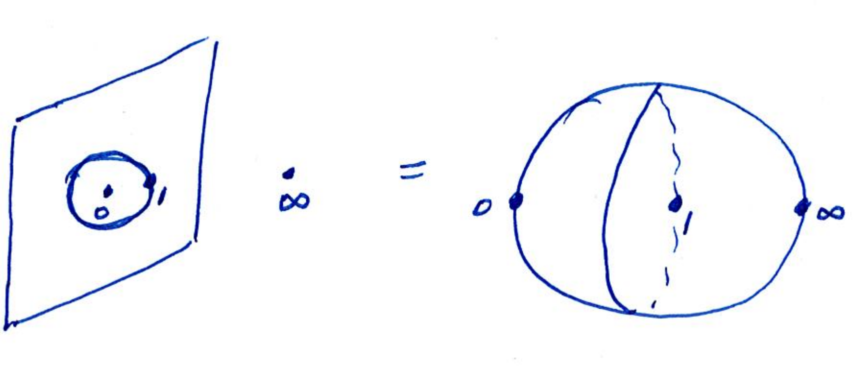
$\pi_+(\pi_B(g)) = g B_{af}$, if $g \in B_{af} x_2$ such that $\pi_B(g) \in M^{w_1, w_2}_{G/B,\, \pi_B(h_1 - h_2)}$

Definition of $M^{w_1,w_2}_{G/P,\tau}$: Let $w_1, w_2 \in W^P$ and $\tau \in H_2(G/P)$.

$M^{w_1,w_2}_{G/P,\tau}$ = the variety of $\phi \in Mor(\mathbb{P}^1, G/P)$ such that $\phi_*([\mathbb{P}^1]) = \tau$, $\phi(\infty) \in B_- w_1 P$, $\phi(0) \in B w_2 P$

The notation reflects

$$\{1\} \subseteq S^1 \subseteq \mathbb{C} \subseteq \mathbb{C} \cup \{\infty\} \subseteq \mathbb{P}^1$$



G = complex reductive algebraic group

$\cup|$

P = <u>parabolic subgroup</u>

$\cup|$

B = <u>Borel subgroup</u> (a minimal parabolic subgroup)

If

$$B = \left\{ \begin{pmatrix} * & \ddots & * \\ 0 & & * \end{pmatrix} \right\} \text{ then } B_- = w_0 B w_0 = \left\{ \begin{pmatrix} * & \ddots & 0 \\ * & & * \end{pmatrix} \right\}$$

The <u>flag variety</u> is G/P and

$$G = \bigsqcup_{w \in W^P} BwP = \bigsqcup_{w \in W^P} B_- wP$$

If $G = GL_n(\mathbb{C})$ then

$W = \{\text{permutation matrices}\} = S_n \subseteq GL_n(\mathbb{C})$,

$W_P = W \cap P$ and $W^P = \{\text{coset reps of cosets in } W/W_P\}$.

If $\phi : \mathbb{P}^1 \longrightarrow G/P$ then we get

$$\phi_* : H_*(\mathbb{P}^1) \longrightarrow H_*(G/P)$$
$$[\mathbb{P}^1] \longmapsto \phi_*([\mathbb{P}^1])$$

(note: $H_*(\mathbb{P}^1) = H_0(\mathbb{P}^1) + H_2(\mathbb{P}^1) = \mathbb{Z}[pt] + \mathbb{Z}[\mathbb{P}^1]$).

$$\pi_P : Q^\vee \longrightarrow H_2(G/P)$$
$$\alpha_i^\vee \longmapsto \begin{cases} [X_{s_i}], & \text{if } s_i \notin W_P \\ 0, & \text{if } s_i \in W_P \end{cases}$$

where $X_{s_i} = Bs_iP \sqcup P \subseteq G/P$.

$$\widetilde{G} = Mor(\mathbb{C}^\times, G)$$

$$\widetilde{G/P} = Mor(\mathbb{C}^\times, G/P) \quad \text{and} \quad \pi_P : \widetilde{G} \longrightarrow \widetilde{G/P}$$
$$g(t) \longmapsto g(t)P$$

Let

$$\widetilde{B} = Mor(\mathbb{C}^\times, B)$$

$$(\widehat{B})_0 = \{\phi \in \widehat{B} \mid \phi|_{S^1} \text{ is trivial on } \pi_1(B)\}$$

$$B_{af} = \{g \in Mor(\mathbb{P}^1 \setminus \{\infty\}, G) \mid g(0) \in B\}$$

$$B_{af}^- = \{g \in Mor(\mathbb{P}^1 \setminus \{0\}, G) \mid g(0) \in B^-\}$$

so that B_{af} is the positive Iwahori

and B_{af}^- is the negative Iwahori.

Then (Theorem 2(B) Lec 11 and beginning of Lec 12)

$$\widehat{G} = \bigsqcup_{x \in W_{af}} B_{af}^- x (\widehat{B})_0 = \bigsqcup_{y \in W_{af}} B_{af}\, y\, (\widehat{B})_0$$

$$= \bigsqcup_{x \in W_{af}} B_{af}^- x B_{af} = \bigsqcup_{y \in W_{af}} B_{af}\, y\, B_{af}.$$

and Proposition 2 of lecture 12 is

$$B_{af}^- x_1 B_{af} \cap B_{af} x_2 B_{af} \underset{\phi_2}{\overset{\phi_1}{\rightleftarrows}} B_{af}^- x_1 (\widehat{B})_0 \cap B_{af} x_2 (\widehat{B})_0$$

$$b^- x_1 B_{af} \longmapsto b^- x_1 (\widehat{B})_0$$

$$b^+ x_2 B_{af} \longleftarrow b^+ x_2 (\widehat{B})_0$$

$$W_{af} = \{ w t_h \mid w \in W,\ h \in Q^\vee \} \quad \text{with}$$

$$t_h t_{h'} = t_{h+h'} \quad \text{and} \quad w t_h w^{-1} = t_{wh}.$$

<u>For $GL_n(\mathbb{C})$:</u>

$$G_{af} = \hat{G} = GL_n(\mathbb{C}[t,t^{-1}]) \quad \text{or} \quad GL_n(\mathbb{C}((t)))$$

$$Q^\vee = \mathbb{Z}\varepsilon_1^\vee + \cdots + \mathbb{Z}\varepsilon_n^\vee = \{ \lambda_1 \varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee \mid \lambda_i \in \mathbb{Z} \}$$

and

$$t_{\lambda_1\varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee} = \begin{pmatrix} t^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & t^{\lambda_n} \end{pmatrix}$$

$$W = \{ w \in GL_n \mid w \text{ is a permutation matrix} \} = S_n$$

$$B_{af} = \left\{ \begin{pmatrix} a_1(t) & & a_{ij}(t) \\ & \ddots & \\ b_{ji}(t) & & a_n(t) \end{pmatrix} \;\middle|\; \begin{matrix} a_i(t) \in \mathbb{C}[[t]]^\times \\ a_{ij}(t) \in \mathbb{C}[[t]] \\ b_{ji}(t) \in t\,\mathbb{C}[[t]] \end{matrix} \right\}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} a_1(t) & & a_{ij}(t) \\ & \ddots & \\ 0 & & a_n(t) \end{pmatrix} \;\middle|\; \begin{matrix} a_i(t) \in \mathbb{C}((t))^\times \\ a_{ij}(t) \in \mathbb{C}((t)) \end{matrix} \right\}$$

$$(\mathcal{B})_0 = \left\{ \begin{pmatrix} a_1 & & a_{ij}(t) \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \;\middle|\; \begin{matrix} a_i \in \mathbb{C}^\times \\ a_{ij}(t) \in \mathbb{C}((t)) \end{matrix} \right\}$$

Peterson Lecture 11: notation

For $SL_2(\mathbb{C})$:

$$\hat{\mathcal{B}} = \left\{ \begin{pmatrix} a(t) & b(t) \\ 0 & d(t) \end{pmatrix} \;\middle|\; a,b,d \in \mathbb{C}[t,t^{-1}],\ ad=1 \right\}$$

$$(\hat{\mathcal{B}})_0 = \left\{ \begin{pmatrix} \lambda & b(t) \\ 0 & \lambda^{-1} \end{pmatrix} \;\middle|\; \lambda \in \mathbb{C}^\times,\ b \in \mathbb{C}[t,t^{-1}] \right\}$$

$$\mathcal{B}_{af} = \left\{ \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{C}[t],\ ad-bc=1,\ c(0)=0 \right\}$$

For $GL_n(\mathbb{C})$:

$$\hat{\mathcal{B}} = \left\{ \begin{pmatrix} a_1(t) & & a_{ij}(t) \\ & \ddots & \\ 0 & & a_n(t) \end{pmatrix} \;\middle|\; \begin{matrix} a_i(t) \in \mathbb{C}[t,t^{-1}]^\times \\ a_{ij}(t) \in \mathbb{C}[t,t^{-1}] \end{matrix} \right\}$$

$$(\hat{\mathcal{B}})_0 = \left\{ \begin{pmatrix} a_1 & & a_{ij}(t) \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \;\middle|\; \begin{matrix} a_i \in \mathbb{C}^\times \\ a_{ij}(t) \in \mathbb{C}[t,t^{-1}] \end{matrix} \right\}$$

$$\mathcal{B}_{af} = \left\{ \begin{pmatrix} a_1(t) & & a_{ij}(t) \\ & \ddots & \\ b_{ji}(t) & & a_n(t) \end{pmatrix} \;\middle|\; \begin{matrix} a_i(t) \in \cancel{\mathbb{C}[t,t^{-1}]}\ \mathbb{C}[t]^\times \\ a_{ij}(t) \in \mathbb{C}[t] \\ b_{ji}(t) \in t\mathbb{C}[t] \end{matrix} \right\}$$

$$W_{af} = \{ w t_h \mid w \in W,\ h \in Q^\vee \} \quad \text{with}$$

$$t_h t_{h'} = t_{h+h'} \quad \text{and} \quad w t_h w^{-1} = t_{wh}.$$

<u>For $GL_n(\mathbb{C})$:</u>

$$Q^\vee = \mathbb{Z}\varepsilon_1^\vee + \cdots + \mathbb{Z}\varepsilon_n^\vee = \{ \lambda_1 \varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee \mid \lambda_i \in \mathbb{Z} \}$$

and
$$t_{\lambda_1 \varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee} = \begin{pmatrix} t^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & t^{\lambda_n} \end{pmatrix}$$