Univ. of Melbourne Number Theory seminar

Heisenberg groups in general 02.03.2015

K is locally compact abelian

$\mathbb{C}_1^\times = \{z \in \mathbb{C} \mid |z| = 1\}$

A <u>central extension</u> is an exact sequence

$1 \to \mathbb{C}_1^\times \xrightarrow{\iota} G \xrightarrow{\pi} K \to 0$ with $\operatorname{im} \iota \subseteq Z(G)$.

Then

$G = \mathbb{C}_1^\times \times K$ with $(\lambda, x)(\mu, y) = (\lambda\mu\psi(x,y), x+y)$

where $\psi: K \times K \to \mathbb{C}^\times$ is such that

$$\psi(x,y)\psi(x+y,z) = \psi(x,y+z)\psi(y,z)$$

Let
$$\begin{aligned} K &\to G \\ x &\mapsto \tilde{x} = (\lambda_x, x) \end{aligned}$$
be a section of $\pi: G \to K$

and define

$e: K \times K \to \mathbb{C}_1^\times$ by $e(x,y) = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$

and
$$\begin{aligned} \varphi: K &\to \hat{K} \\ x &\mapsto \varphi_x: K \to \mathbb{C}_1^\times \\ &\qquad y \mapsto e(x,y) \end{aligned}$$
where $\hat{K} = \operatorname{Hom}(K, \mathbb{C}_1^\times)$.

<u>Definition</u> G is a <u>Heisenberg group</u> if

$\varphi: K \to \hat{K}$ is an isomorphism.

<u>HW</u>: Show that e is multiplicative, e is skew, and e is a pairing, i.e.

$$e(x+x',y)=e(x,y)\,e(x',y), \qquad e(x,x)=1,$$

$$e(x,y+y')=e(x,y)\,e(x,y'), \qquad e(x,y)=e(y,x)^{-1},$$

and

$$e(x,y)=\frac{\psi(x,y)}{\psi(y,x)}.$$

The example $G = Heis(2g, \mathbb{R})$

$K = V = \mathbb{R}^{2g}$ and $1 \to \mathbb{C}_1^\times \to Heis(V) \to V \to 0$

Then $\psi : V\times V \to \mathbb{C}^\times$ and $e : V\times V \to \mathbb{C}^\times$ can be given by

$$e(x,y) = e^{2\pi i A(x,y)} \quad \text{and} \quad \psi(x,y) = e^{2\pi i \frac{1}{2} A(x,y)}$$

where
$A : V\times V \to \mathbb{R}$ is a nondegenerate skewsymmetric $\mathbb{R}$-bilinear form.

We may choose an $\mathbb{R}$-basis $e_1^{(1)}, \ldots, e_g^{(1)}, e_1^{(2)}, \ldots, e_g^{(2)}$ of $V = \mathbb{R}^{2g}$ so that

A has matrix $\left(\begin{array}{c|c} 0 & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} \\ \hline \begin{smallmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{smallmatrix} & 0 \end{array}\right)$ and

$$A(x,y) = (x_1^t, x_2^t)\left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (-x_2^t, x_1^t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1^t y_2 - x_2^t y_1$$

Then

$$G = Heis(2g, \mathbb{R}) = \{(\lambda, x) \mid \lambda \in \mathbb{C}_1^\times, x \in \mathbb{R}^{2g}\} \quad \text{with}$$

$$(\lambda, x)(\mu, y) = (\lambda\mu e^{2\pi i \frac{1}{2} A(x,y)}, x+y).$$

Then $Lie(G)$ has basis $\{\underbrace{p_1, \ldots, p_g}_{\substack{\text{momentum} \\ \text{operators}}}, \underbrace{q_1, \ldots, q_g}_{\substack{\text{position} \\ \text{operators}}}, \hbar\}$ with

$$[p_i, p_j] = 0, \qquad [q_i, q_j] = 0, \qquad \hbar \in Z(Lie(G))$$

$$[p_i, q_j] = \delta_{ij}\hbar$$

The Stone-von Neumann Theorem   General G.

An *isotropic subgroup* H of K is a closed subgroup $H \subseteq K$ such that $e|_{H \times H} = 1$.

Theorem 1.2 Let G be a Heisenberg group.

Choose $H \subseteq K$ a maximal isotropic subgroup and

$$\begin{aligned} \sigma : H &\to G \\ h &\mapsto (\alpha(h), h) \end{aligned}$$

a homomorphism with $\pi \circ \sigma = id_H$

Let

$$L^2(K//H) = \left\{ f : K \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is measurable} \\ f(x+h) = \alpha(h)^{-1} \psi(h,x)^{-1} f(x) \text{ for } h \in H \\ \int_{K/H} |f(x)|^2 dx < \infty \end{array} \right\}$$

with G-action given by

$$(U_{(\lambda,y)} f)(x) = \lambda \psi(x,y) f(x+y)$$

Then $L^2(K//H)$ is the unique irreducible unitary G-module such that

$$U_{(\lambda,0)} = \lambda \cdot id \quad \text{for } \lambda \in \mathbb{C}_1^\times .$$

Choices of maximal isotropic subgroups for $K = V = \mathbb{R}^{2g}$

Example 1: $W_2 = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \right\} \subseteq \mathbb{R}^{2g}$, which gives

$$L^2(\mathbb{R}^g) = \left\{ \varphi : \mathbb{R}^g \to \mathbb{C} \,\middle|\, \int |\varphi(x_1)|^2 dx_1 < \infty \right\}$$

with $G = \mathrm{Heis}(2g, \mathbb{R})$ action

$$(U_{(\lambda, y_1, y_2)} \varphi)(x_1) = \lambda e^{2\pi i (x_1^t y_2 + \frac{1}{2} y_1^t y_2)} \varphi(x_1 + y_1)$$

This representation has differential $d\rho_1 : \mathrm{Lie}(G) \to \mathrm{End}(L^2(\mathbb{R}^g))$ given by

$$(p_i f)(x) = \left(\frac{\partial}{\partial x_i} f\right)(x), \qquad (q_j f)(x) = 2\pi i x_j f(x)$$

$$(\hbar f)(x) = 2\pi i f(x).$$

Example 2: $L = \mathbb{Z}^{2g}$ with $\sigma : L \to \mathrm{Heis}(2g, \mathbb{R})$

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \longmapsto \left(e^{2\pi i (2 n_1^t n_2)}, \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \right)$$

which gives

$$L^2(\mathbb{R}^{2g}/\mathbb{Z}^{2g}) = \left\{ f : \mathbb{R}^{2g} \to \mathbb{C} \,\middle|\, \begin{array}{l} \int_{\mathbb{R}^{2g}/\mathbb{Z}^{2g}} |f(x)|^2 < \infty, \text{ and for } n \in \mathbb{Z}^{2g} \\ f(x+n) = e^{2\pi i (n_1^t n_2)} e^{-i\pi A(n,x)} f(x) \end{array} \right\}$$

with $G = \mathrm{Heis}(2g, \mathbb{R})$-action

$$(U_{(\lambda, y)} f)(x) = \lambda e^{i\pi A(x,y)} f(x+y)$$

As $\mathrm{Heis}(2g, \mathbb{R})$-modules,

$$L^2(\mathbb{R}^g) \xrightarrow{\sim} L^2(\mathbb{R}^{2g}/\mathbb{Z}^{2g})$$
$$f \longmapsto f^*$$

where

$$f^*\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sum_{n \in \mathbb{Z}^g} f(x_1 + n)\, e^{2\pi i (n^t x_2 + \frac{1}{2} x_1^t x_2)}$$

and

$$f(x_1) = \int_{\mathbb{R}^g/\mathbb{Z}^g} f^*\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} e^{2\pi i (-x_1^t x_2 \frac{1}{2})} \, dx_2 .$$

Fock space in two versions

The Siegel upper half space is

$$\mathfrak{S}_g = \{ \tau \in M_{g\times g}(\mathbb{C}) \mid \tau^t = \tau \text{ and } \operatorname{Im}\tau \text{ is pos. definite} \}.$$

Fix $\tau \in \mathfrak{S}_g$. Define

$$\mathcal{H}^2_\Theta(\mathbb{C}^g, \tau) = \left\{ f : \mathbb{C}^g \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is holomorphic} \\ \|f\|^2 < \infty \end{array} \right\}$$

where

$$\|f\|^2 = \int_{\mathbb{C}^g} |f(x)|^2 e^{-2\pi x_1^t \cdot \operatorname{Im}\tau \cdot x_1} \, dx_1\, dx_2$$

with $G = \operatorname{Heis}(2g, \mathbb{R})$ action given by

$$(U_{(\lambda,y)} f)(\underline{x}) = \lambda^{-1} e^{2\pi i (y_1^t \cdot \underline{x} + \frac{1}{2} y_1^t \cdot y)} f(\underline{x} + y)$$

Let

$$\mathcal{H}^2_\phi(\mathbb{C}^g, \tau) = \left\{ f : \mathbb{C}^g \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is holomorphic} \\ \|f\|^2 < \infty \end{array} \right\}$$

where
$$\|f\|^2 = \int_{\mathbb{C}^g} |f(x)|^2 e^{-\pi H(\underline{x},\underline{x})} \, d\underline{x} < \infty, \quad H(\underline{x},\underline{x}) = \underline{x}^t \cdot (\operatorname{Im}\tau)^{-1} \cdot \overline{\underline{x}},$$

and the $G = \operatorname{Heis}(2g, \mathbb{R})$ action is given by

$$(U_{(\lambda,y)} f)(\underline{x}) = \lambda^{-1} e^{-\pi H(\underline{x},y) - \frac{\pi}{2} H(y,y)} f(\underline{x} + y)$$

Then

$$\mathcal{H}^2_\Theta(\mathbb{C}^g, \tau) \xrightarrow{\sim} \mathcal{H}_\phi(\mathbb{C}^g, \tau)$$

$$f(\underline{x}) \longmapsto e^{\frac{\pi}{2} \underline{x}^t \cdot (\operatorname{Im}\tau)^{-1} \cdot \underline{x}} f(\underline{x})$$

as $G = \operatorname{Heis}(2g, \mathbb{R})$-modules.

The differential of the G representation on $\mathcal{H}_\phi(\mathbb{C}^g,\tau)$ is

$$d\rho_\tau : \mathrm{Lie}(G) \longrightarrow \mathrm{End}(\mathcal{H}_\phi(\mathbb{C}^g,\tau)) \quad \text{given by}$$

$$(p_i f)(\underline{x}) = \left(-\pi \sum_k (\bar{\tau}(\mathrm{Im}\,\tau)^{-1})_{ik}\underline{x}_k + \sum_j \tau_{ij}\frac{\partial}{\partial \underline{x}_j}\right) f$$

$$(q_j f)(\underline{x}) = \left(-\pi \sum_k ((\mathrm{Im}\,\tau)^{-1})_{jk}\underline{x}_k + \frac{\partial}{\partial \underline{x}_j}\right) f.$$

For $\tau \in \mathfrak{G}_g$ let

$$W_\tau = \mathbb{C}\,\mathrm{span}\{p_i - \sum_j \tau_{ij} q_j \mid i = 1,2,\ldots,n\}$$

$$W_{\bar{\tau}} = \mathbb{C}\,\mathrm{span}\{p_i - \sum_j \bar{\tau}_{ij} q_j \mid i = 1,2,\ldots,n\}.$$

<u>Theorem</u> <u>2.2</u>

(a) In $L^2(\mathbb{R}^g)$,

$f_\tau(x_1) = e^{i\pi x_1^t \tau x_1}$ is the unique (up to constants) vector killed by W_τ.

(b) In $L^2(\mathbb{R}^{2g}/\mathbb{Z}^{2g})$,

$f_\tau\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{i\pi x_1^t \cdot \underline{x}}\,\theta(\underline{x},\tau)$, with $\underline{x} = \tau x_1 + x_2$.

is the unique (up to constant multiples) vector killed by W_τ.

(c) In $\mathcal{H}_\phi^2(\mathbb{C}^g,\tau)$,

$f_\tau = 1$, is the unique (up to constant multiples) vector killed by $W_{\bar{\tau}}$.