Two boundary Hecke and Temperley-Lieb algebras

Two boundary Temperley Lieb

Pure Math. Seminar
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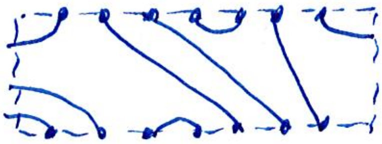
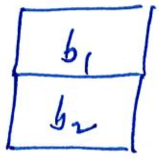
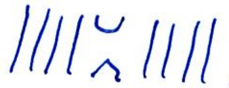
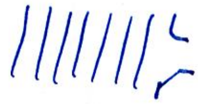
(0)

noncrossing diagram 

multiplication  $= b_1 b_2$

More precisely, TL_k has generators

$e_0 =$ , $e_i =$ , $e_k =$ 

and relations

$$e_0^2 = L e_0, \quad e_i^2 = (q+q^{-1})e_i, \quad e_k^2 = R e_k$$

and

$e_ie_j = e_je_i$ for $j = i\pm 1$, $e_ie_{i\pm1}e_i = e_i$ for $i = 1, 2, \ldots, k-1$.

De Gier-Nichols studied TL_k-modules, in particular

$W^{(k)}(z) = TL_k \cdot d_f$ where $d_f =$  and  $= z$

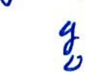
and $W^{(k,j)}_{\epsilon_L,\epsilon_R}(z) = \mathrm{span}\{\text{diagrams with } j\text{-through strands}\}$
$= TL_k d_j$ (where d_j is a specific diagram with j-through strands).

OUR POINT: TL_k is a quotient of the two boundary Hecke algebra H_k^{ext}
and one can use affine Hecke algebra representation theory to answer questions about $W^{(k)}(z)$ and $W^{(k,j)}_{\epsilon_L,\epsilon_R}$.

The two boundary braid group $\mathcal{B}_k$

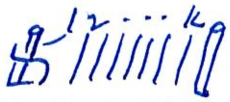
Notation: 

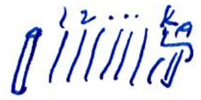
means $ghgh = hghg$

means $ghg = hgh$, 

means $gh = hg$.

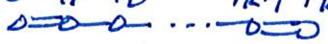
$\mathcal{B}_k$ is given by generators and relations:

$T_0 =$ 

$T_k =$ 

$T_i =$ 

$T_0 \quad T_1 \quad T_2 \quad \dots \quad T_{k-1} \quad T_k$



$\Longleftrightarrow$

$X_1 =$ 

$Y_1 =$ 

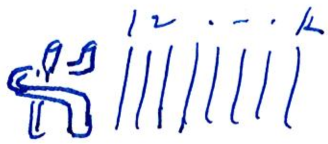
$T_i =$ 

$X_1 \quad T_1 \quad T_2 \qquad T_{k-1}$

$Y_1 \quad T_1 \quad T_2 \qquad T_{k-1}$

$$Y_1(T_1X_1T_1^{-1}) = (T_1X_1T_1^{-1})Y_1$$

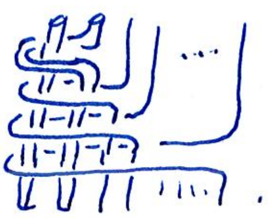
$\mathcal{B}_k^{ext}$ also has

$P =$ 

Then

$$\mathcal{B}_k^{ext} = \mathcal{B}_k \times \mathcal{D}, \text{ where } \mathcal{D} = \{Z_0^j \mid j \in \mathbb{Z}\} \cong \mathbb{Z}$$

with

$Z_0 =$ 
.

The two boundary Hecke algebra H_k^{ext}

H_k^{ext} is $\mathbb{C}\mathcal{B}_k^{ext}$ with additional relations

$$(X_1-a_1)(X_1-a_2)=0, \quad (Y_1-b_1)(Y_1-b_2)=0, \quad (T_i-t^{\frac{1}{2}})(T_i+t^{-\frac{1}{2}})=0$$

Let $W_0 = (-1)^k (a_1a_2b_1b_2)^{-\frac{1}{2}} Z_0$ and, for $i=1,2,\ldots,k$,

$W_i = -(a_1a_2b_1b_2)^{-\frac{1}{2}} Z_i$ with $Z_i =$ 

The Weyl group of type C has generators and relations


and $s_i^2=1$ for $i=0,1,\ldots,k-1$.

$\mathcal{W}_0$ acts on sequences $(\lambda_0,\lambda_1,\ldots,\lambda_k)\in\mathbb{Z}^{k+1}$ by

$$s_0(\lambda_0,\lambda_1,\ldots,\lambda_k) = (\lambda_0,-\lambda_1,\lambda_2,\ldots,\lambda_k)$$

$$s_i(\lambda_0,\lambda_1,\ldots,\lambda_k) = (\lambda_0,\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\lambda_i,\lambda_{i+2},\ldots,\lambda_k)$$

Let

$$W^\lambda = W_0^{\lambda_0}W_1^{\lambda_1}\cdots W_k^{\lambda_k}, \quad t_0^{\frac{1}{2}} = b_1^{\frac{1}{2}}(-b_2)^{-\frac{1}{2}}, \quad t_k^{\frac{1}{2}} = a_1^{\frac{1}{2}}(-a_2)^{-\frac{1}{2}}$$

Then

$$T_iW^\lambda = W^{s_i\lambda}T_i + (t^{\frac{1}{2}}-t^{-\frac{1}{2}})\frac{W^\lambda - W^{s_i\lambda}}{1-W_iW_{i+1}^{-1}} \quad \text{for } i=1,2,\ldots,k-1$$

$$T_0W^\lambda = W^{s_0\lambda}T_0 + \left((t_0^{\frac{1}{2}}-t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}}-t_k^{-\frac{1}{2}})W_1^{-1}\right)\frac{W^\lambda - W^{s_0\lambda}}{1-W_1^{-2}}.$$

i.e. H_k^{ext} is an affine Hecke algebra of type C_k (see Lusztig JAMS 1989).

An H_k^{ext}-module M is <u>calibrated</u> if

$$M = \bigoplus_\gamma M_\gamma$$

where, for $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_k) \in (\mathbb{C}^\times)^{k+1}$,

$$M_\gamma = \{ m \in M \mid W_i m = \gamma_i m \text{ for } i = 0, 1, \ldots, k \}.$$

<u>Theorem</u> The irreducible calibrated H_k^{ext}-modules are

$$H_k^{(\gamma,J)} = \mathrm{span}\{ v_w \mid w \in \mathcal{F}^{(\gamma,J)} \} \quad \text{with}$$

$$W_i v_w = \gamma_{w^{-1}(i)} v_w,$$

$$T_{s_i} v_w = \left(\frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - \gamma_{w^{-1}(i)} \gamma^{-1}_{w^{-1}(i+1)}} \right) v_w + \sqrt{\cdots}\; v_{s_i w},$$

$$T_{s_0} v_w = \left(\frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-1/2}) \gamma^{-1}_{w^{-1}(1)}}{1 - \gamma^{-2}_{w^{-1}(1)}} \right) v_w + \sqrt{\cdots}\; v_{s_0 w}$$

where $v_{s_i w} = 0$ if $s_i w \notin \mathcal{F}^{(\gamma,J)}$.

Here

$\gamma \in (\mathbb{C}^\times)^{k+1}$ and $J \subseteq \{$pos. roots of a type C_k rootsystem$\}$

<u>POINT</u>: We can explicitly construct all all irreducible calibrated H_k^{ext}-modules

Schur-Weyl duality

Fix irreducible $U_q\mathfrak{gl}_n$-modules

$$M = L\Big(c\big\{\overbrace{\square\square}^{a}\Big), \qquad N = L\Big(d\big\{\overbrace{\square}^{b}\Big), \qquad V = L(\square).$$

Then H_k^{ext} with

$$t^{\frac{1}{2}} = q, \quad a_1 = q^{2a}, \quad a_2 = q^{-2c}, \quad b_1 = q^{2b}, \quad b_2 = q^{-2d}$$

acts on $M \otimes N \otimes V^{\otimes k}$,

$M \otimes N \otimes V \otimes V \otimes V \otimes V$

$M \otimes N \otimes V \otimes V \otimes V \otimes V$

(R-matrices)

and the $U_q\mathfrak{gl}_n$ and H_k^{ext}-actions commute. So

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda} L(\lambda) \otimes H_k^{\lambda}$$

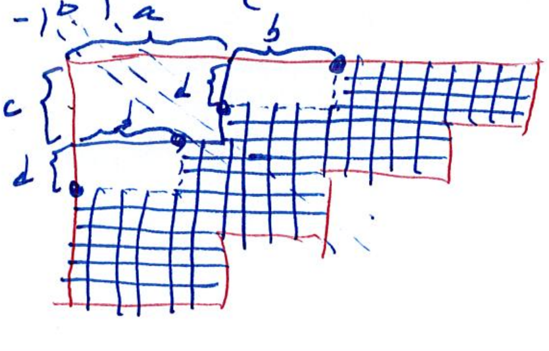
simple $U_q\mathfrak{gl}_n$-mod.

Theorem H_k^{λ} is an irreducible calibrated H_k^{ext}-module,

$$H_k^{\lambda} \cong H_k^{(\gamma, J)} \quad \text{where } J = \dots$$

and

$$\{\gamma_1, \dots, \gamma_k\} = \{q^{\text{shifted content(box)}} \mid \text{box} \in \text{skew shape}\}$$



shifted content = diagonal number.