Theta functions as matrix coefficients for Heisenberg representations

Number theory Seminar, University of Melbourne, 16 March 2015 ①

A. Ram

The group $Heis(2g, \mathbb{R})$ and its Lie algebra

$$G = Heis(2g, \mathbb{R}) = \{(\lambda, x) \mid \lambda \in \mathbb{C}_1^\times, x \in \mathbb{R}^{2g}\}$$

with

$$(\lambda, x)(\mu, y) = (\lambda\mu e^{2\pi i \frac{1}{2} A(x,y)}, x+y) \quad \text{where}$$

$\mathbb{C}_1^\times = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and

$A: \mathbb{R}^{2g} \times \mathbb{R}^{2g} \to \mathbb{R}$ is a nondeg. skew symm. bilinear form.

$Lie(G)$ has basis $\{p_1, \ldots, p_g, q_1, \ldots, q_g\}$ with

$$[p_i, p_j] = 0, \quad [q_i, q_j] = 0, \quad [h, p_i] = 0, \quad [h, q_j] = 0$$

$$[p_i, q_j] = \delta_{ij} h.$$

The module $\mathcal{H}$ version 1

$$L^2(\mathbb{R}^g) = \{f: \mathbb{R}^g \to \mathbb{C} \mid \int |f(x_1)|^2 dx_1 < \infty\}$$

with

$$(U_{(\lambda, y_1, y_2)} f)(x_1) = \lambda e^{2\pi i (x_1^t y_2 + \frac{1}{2} y_1^t y_2)} f(x_1 + y_1)$$

$$(p_i f)(x_1) = \left(\frac{\partial}{\partial x_i} f\right)(x_1), \quad (q_j f)(x_1) = 2\pi i x_j f(x_1)$$

$$(h f)(x_1) = 2\pi i f(x_1).$$

The module $\mathcal{H}$ version 2

$$L^2(\mathbb{R}^{2g}/\mathbb{Z}^{2g}) = \left\{ f : \mathbb{R}^{2g} \to \mathbb{C} \;\middle|\; \begin{array}{l} \int_{\mathbb{R}^{2g}/\mathbb{Z}^{2g}} |f(x)|^2 < \infty, \text{ and for } n \in \mathbb{Z}^{2g} \\ f(x+n) = e^{2\pi i\, n_1^t n_2} e^{-i\pi A(n,x)} f(x) \end{array} \right\}$$

with $(U_{(\lambda,y)} f)(x) = \lambda e^{i\pi A(x,y)} f(x+y)$

The module $\mathcal{H}$ version 3 (really the module $\mathcal{H}^*$)

$$\mathcal{H}_\Theta^2(\mathbb{C}^g, \tau) = \left\{ f : \mathbb{C}^g \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is holomorphic and} \\ \int_{\mathbb{C}^g} |f(\underline{x})|^2 e^{-2\pi x_1^t \cdot \mathrm{Im}\,\tau \cdot x_1} dx_1\, dx_2 \end{array} \right\}$$

with $(U_{(\lambda,y)} f)(\underline{x}) = \lambda^{-1} e^{2\pi i (y_1^t \cdot \underline{x} + \frac{1}{2} y_1^t \cdot \underline{x})} f(\underline{x}+\underline{y})$

The module $\mathcal{H}$ version 4

$$\mathcal{H}_\Phi^2(\mathbb{C}^g, \tau) = \left\{ f : \mathbb{C}^g \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is holomorphic and} \\ \int_{\mathbb{C}^g} |f(\underline{x})|^2 e^{-\pi \underline{x}^t (\mathrm{Im}\,\tau)^{-1} \overline{\underline{x}}} d\underline{x} \end{array} \right\}$$

with $(U_{(\lambda,y)} f)(\underline{x}) = \lambda^{-1} e^{-\pi (\underline{x} - \frac{1}{2}\underline{y})^t (\mathrm{Im}\,\tau)^{-1} \overline{\underline{x}}} f(\underline{x}+\underline{y})$

$$(p_i f)(\underline{x}) = \left(-\pi \sum_k (\overline{\tau} (\mathrm{Im}\,\tau)^{-1})_{ik} \underline{x}_k + \sum_j \tau_{ij} \frac{\partial}{\partial \underline{x}_j} \right) f$$

$$(q_j f)(\underline{x}) = \left(-\pi \sum_k ((\mathrm{Im}\,\tau)^{-1})_{jk} \underline{x}_k + \frac{\partial}{\partial \underline{x}_j} \right) f$$

Distinguished vectors

$\sigma : \mathbb{Z}^{2g} \to Heis(2g, \mathbb{R})$ is given by

$$\sigma(n_1, n_2) = (e^{2\pi i \frac{1}{2} n_1^t n_2}, n_1, n_2)$$

For $\tau \in \mathfrak{S}_g$, $\left(\mathfrak{S}_g = \{ \tau \in M_{g\times g}(\mathbb{C}) \mid \tau^t = \tau,\ Im\,\tau \text{ is pos. def.} \} \right)$

$$W_\tau = \operatorname{span}\{ p_i - \sum_j \tau_{ij} q_j \mid i = 1, 2, \ldots, g \}$$

$$W_{\bar\tau} = \operatorname{span}\{ p_i - \sum_j \bar\tau_{ij} q_j \mid i = 1, 2, \ldots, g \}$$

and define $\theta : \mathbb{C}^g \to \mathbb{C}$ by

$$\theta(\underline{x}, \tau) = \sum_{n \in \mathbb{Z}^{2g}} e^{i\pi n^t \tau n + 2\pi i n^t \cdot \underline{x}}$$

Mumford Theorems 2.2 and 2.3

(a) $L^2(\mathbb{R}^g)^{W_\tau} = \operatorname{span}\{ e^{i\pi x_1^t \tau x_1} \}$ and $\mathcal{H}^2_\phi(\mathbb{C}^g, \tau)^{W_{\bar\tau}} = \operatorname{span}\{1\}$

(b) $\left(L^2(\mathbb{R}^{2g}/\mathbb{Z}^{2g})_{-\infty} \right)^{\sigma(\mathbb{Z}^{2g})} = \operatorname{span}\left\{ \sum_{n \in \mathbb{Z}^{2g}} e^{2\pi i \frac{1}{2} n_1^t n_2} \delta_n \right\}$ and

$$\left(\mathcal{H}^2_\theta(\mathbb{C}^g, \tau)_{-\infty} \right)^{\sigma(\mathbb{Z}^{2g})} = \operatorname{span}\{ \theta(\underline{x}, \tau) \}.$$

Heis$(2g, \mathbb{R})$-module isomorphisms

$$L^2(\mathbb{R}^g) \longrightarrow L^2(\mathbb{R}^{2g}/\mathbb{Z}^{2g}) \xrightarrow[\sim]{\text{conj. linear}} \mathcal{H}_0^2(\mathbb{C}^g, \tau) \xrightarrow{\sim} \mathcal{H}_\phi^2(\mathbb{C}^g, \tau)$$

$$f(x_1) \longmapsto f^*\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad f(\underline{x}) \longmapsto e^{\frac{1}{2}\pi \underline{x}^t (\mathrm{Im}\,\tau)^{-1} \underline{x}} f(\underline{x})$$

$$e^{i\pi x_1^t \tau x_1} \mapsto e^{i\pi x_1^t \cdot x} \quad \theta(\underline{x}, \tau) \longmapsto e^{-\frac{1}{2} \underline{x}^t (\mathrm{Im}\,\tau)^{-1} \underline{x}} \longmapsto 1$$

$$\sum_{n \in \mathbb{Z}^g} \delta_n \longmapsto \sum_{n \in \mathbb{Z}^{2g}} e^{2\pi i n_1^t n_2} \delta_n \longmapsto \theta(\underline{x}, \tau) \longmapsto e^{\frac{1}{2}\pi \underline{x}^t (\mathrm{Im}\,\tau)^{-1} \underline{x}} \theta(\underline{x}, \tau)$$

where

$$f^*\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sum_{n \in \mathbb{Z}^g} f(x_1 + n) e^{2\pi i (n^t x_2 + \frac{1}{2} x_1^t x_2)}$$

and

$$f(x_1) = \int_{\mathbb{R}^{2g}/\mathbb{Z}^{2g}} f^*\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} e^{2\pi i (-\frac{1}{2} x_1^t x_2)} \, dx_2$$

What is $\mathcal{H}_{-\infty}$?

$\mathcal{H}_\phi^2(\mathbb{C}^g, \tau) \supseteq V_1 \supseteq V_2 \supseteq \cdots$, where

$V_n = \{\text{polynomials in } \underline{x} \text{ of degree} \le n\}$

$= \operatorname{span}\{d_1 d_2 \cdots d_n \cdot 1 \mid d_1, d_2, \ldots, d_n \in \mathrm{Lie}(G)\}$

Then (see Mumford p. 21)

$\mathcal{H}_\infty = \{f \in \mathcal{H} \mid d_1 d_2 \cdots d_n f \text{ "is defined" for } n \in \mathbb{Z}_{\ge 0}, d_1, \ldots, d_n \in \mathrm{Lie}(G)\}$

(Theorem $\overline{\mathcal{H}_\infty} = \mathcal{H}$). Then "$\mathcal{H}_{-\infty}$ is the "graded dual" of $\mathcal{H}_\infty$" and

$\mathcal{H} \subseteq \mathcal{H}_{-\infty} = \{\text{conjugate linear continuous } l : \mathcal{H}_\infty \to \mathbb{C}\}$

$x \longmapsto l_x : \mathcal{H}_\infty \to \mathbb{C}$

$\qquad y \longmapsto \langle x, y \rangle$

Matrix coefficients

There are commuting actions of $Heis(2g,\mathbb{R})$ on $\{f:\mathbb{R}^{2g}\to\mathbb{C}\}$ given by

$$(U^{\text{left}}_{(\lambda,y)}f)(x)=\lambda^{-1}e^{2\pi i\frac{1}{2}A(x,y)}f(x-y)$$

$$(U^{\text{right}}_{(\lambda,y)}f)(y)=\lambda e^{2\pi i\frac{1}{2}A(x,y)}f(x+y)$$

As $Heis(2g,\mathbb{R})$ bimodules

$$L^2(\mathbb{R}^{2g}) \text{ "=" } \mathcal{H}^*\otimes\mathcal{H} \text{ "=" } \operatorname{span}\left\{\begin{array}{l} e_{fg}:\mathbb{R}^{2g}\to\mathbb{C} \\ x\mapsto\langle U_{(1,x)}f,g\rangle\end{array}\middle|\begin{array}{l} f\in\mathcal{H}_\infty \\ g\in\mathcal{H}_{-\infty}\end{array}\right\}$$

(see Mumford p. 32 for more precise discussion of "=").

Mumford Cor 2.4 The function in $\mathcal{H}=L^2(\mathbb{R}^{2g}/\mathbb{Z}^{2g})$ denoted

$$\theta^*\begin{bmatrix}x_1\\x_2\end{bmatrix}(\tau)=e^{-i\pi x_1{}^t x_2}\theta\begin{bmatrix}x_1\\x_2\end{bmatrix}(0;\tau)=e^{i\pi x_1{}^t(\tau x_1+x_2)}\theta(\tau x_1+x_2,\tau)$$

is equal to

$$\langle U_{(1,x)}f_\tau, e_{\mathbb{Z}}\rangle$$

where $f_\tau\in(\mathcal{H}_\infty)^{W_\tau}$ and $e_{\mathbb{Z}}\in(\mathcal{H}_{-\infty})^{\sigma(\mathbb{Z}^{2g})}$.

(Mumford proof p. 27 uses, in $L^2(\mathbb{R}^{2g}/\mathbb{Z}^{2g})_{-\infty}$,

$$e_{\mathbb{Z}}=\sum_{n\in\mathbb{Z}^{2g}}e^{2\pi i n_1{}^t n_2}\delta_n \quad\text{and}\quad f_\tau=e^{i\pi x_1{}^t\cdot \underline{x}}\theta(\underline{x},\tau)$$
)