Picturing representation rings

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Geometric and Categorical
Representation Theory
Mooloolaba 16 December 2015

Formula (6.5.2) from Kac, Infinite dim. Lie algebras

$$t_\beta \lambda = \lambda + m\beta - (\bar{\lambda} + \tfrac{1}{2} m\beta | \beta)\, \delta$$

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$$\mathfrak{h}^* = \mathbb{C}\delta \oplus \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$$

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$$t_\beta \lambda = \lambda + m\beta - \left((\bar{\lambda} + \tfrac{1}{2} m\beta | \beta) \right) \delta \qquad \text{Here } \lambda \in \mathfrak{h}^*.$$

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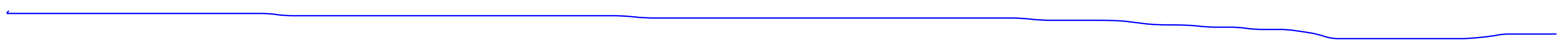
For sl_2, $\mathring{\mathfrak{h}}^* = \text{span}\{\omega_1\}$



Formula (6.5.2) from Kac, Infinite dim. Lie algebras

$$t_\beta \lambda = \lambda + m\beta - \left(\bar{\lambda} + \tfrac{1}{2} m\beta \middle| \beta\right)\delta \qquad \text{Here } \lambda \in \mathfrak{h}^*$$

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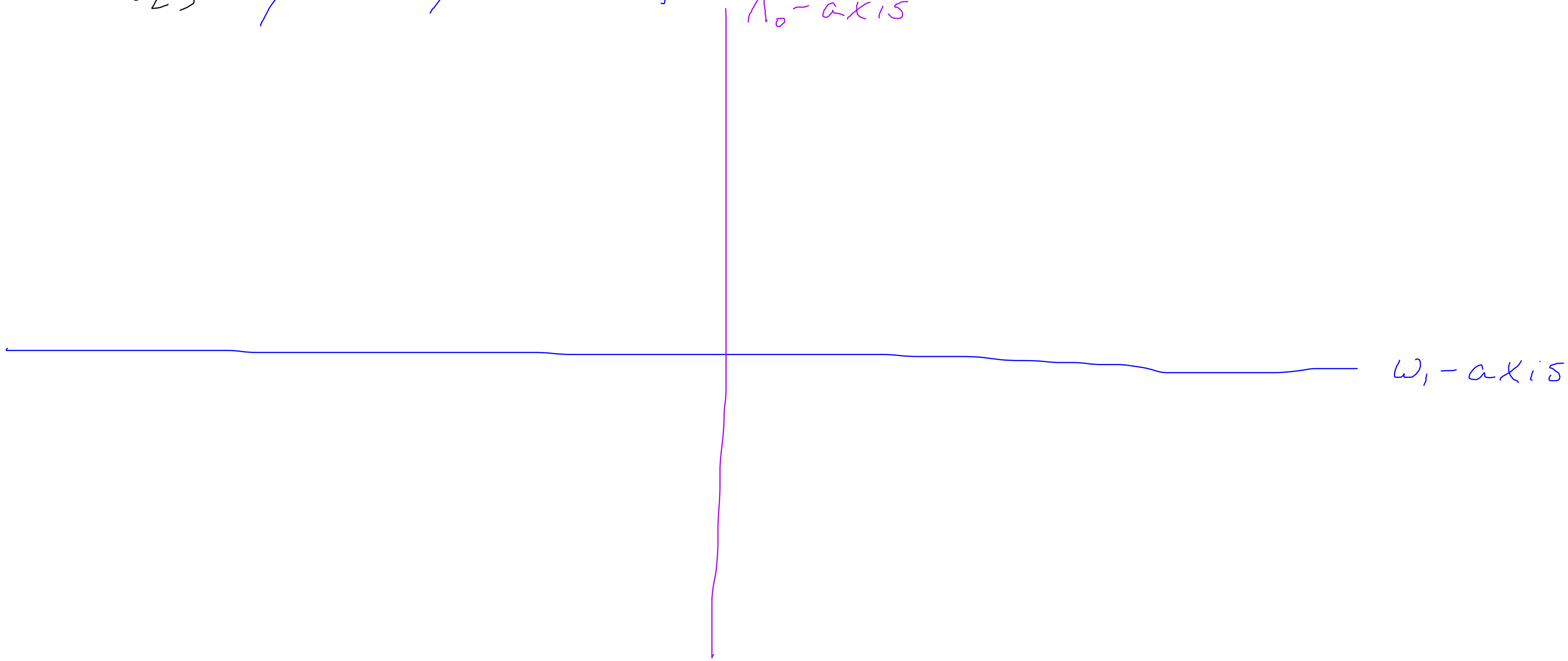

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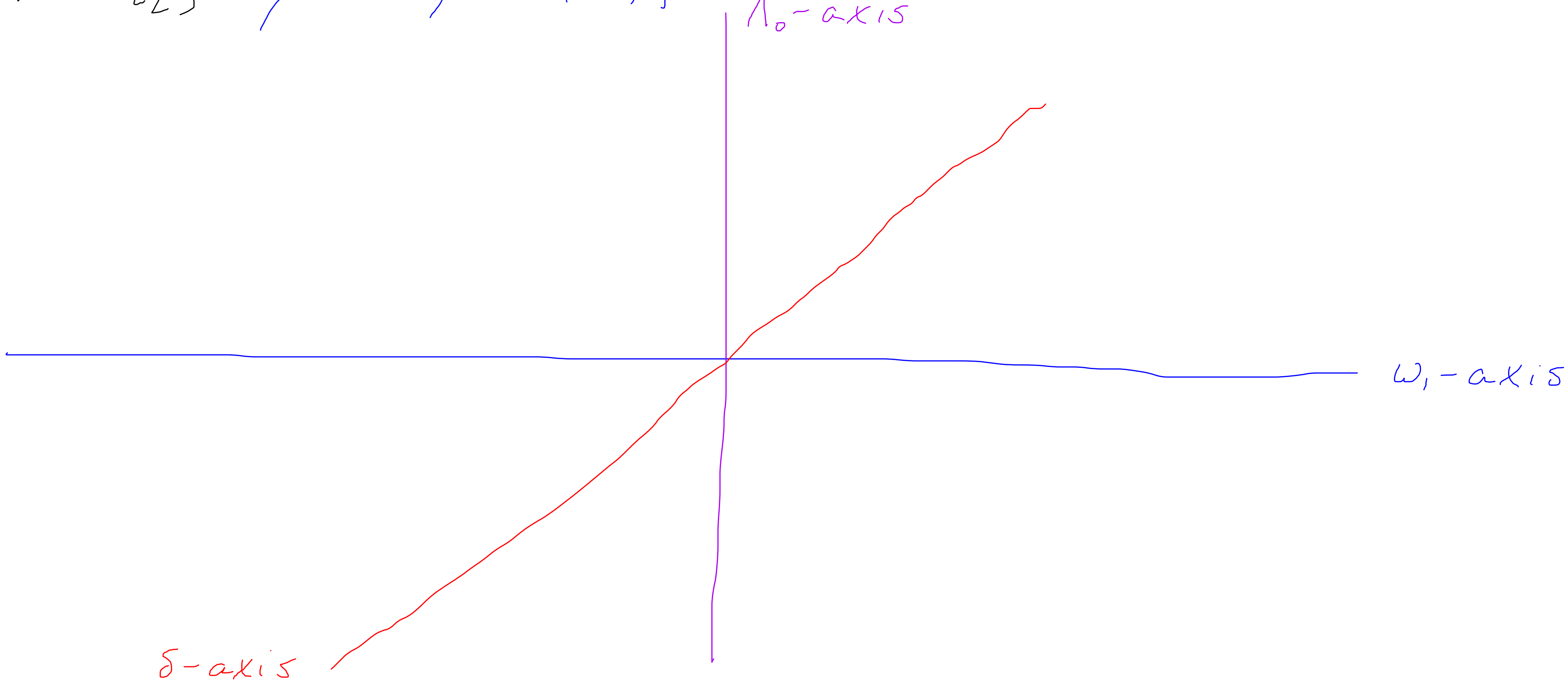

The double affine Weyl group

$$\tilde{W} = \{ q^z X^\mu w Y^\lambda \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}},\ \lambda \in \mathfrak{h}_{\mathbb{Z}},\ w \in W_0,\ z \in \mathbb{C} \}$$

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with

$$q^z X^\mu w Y^\lambda = \left(\begin{array}{c|c|c} 1 & \lambda & z \\ \hline 0 & w & \mu \\ \hline 0 & 0 & 1 \end{array}\right)$$

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$$\mathfrak{h}^* = \mathbb{C}\delta \oplus \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 = \left\{ \left(\begin{array}{c} a \\ \hline \vdots \\ \gamma \\ \vdots \\ \hline \ell \end{array}\right) \,\middle|\, a, \ell \in \mathbb{C},\ \gamma \in \mathring{\mathfrak{h}}^* \right\}$$

$$\widetilde{W} = \{ q^z X^\mu w Y^\lambda \mid \mu \in \mathbb{Z}^{*}_{0}, \lambda \in \mathbb{Z}_{0}, w \in W_0, z \in \mathbb{C} \}$$

$$\widehat{W} = \{ q^z X^\mu w Y^\lambda \mid \mu \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}},\ \lambda \in \mathring{\mathfrak{h}}_{\mathbb{Z}},\ w \in W_0,\ z \in \mathbb{C} \}$$

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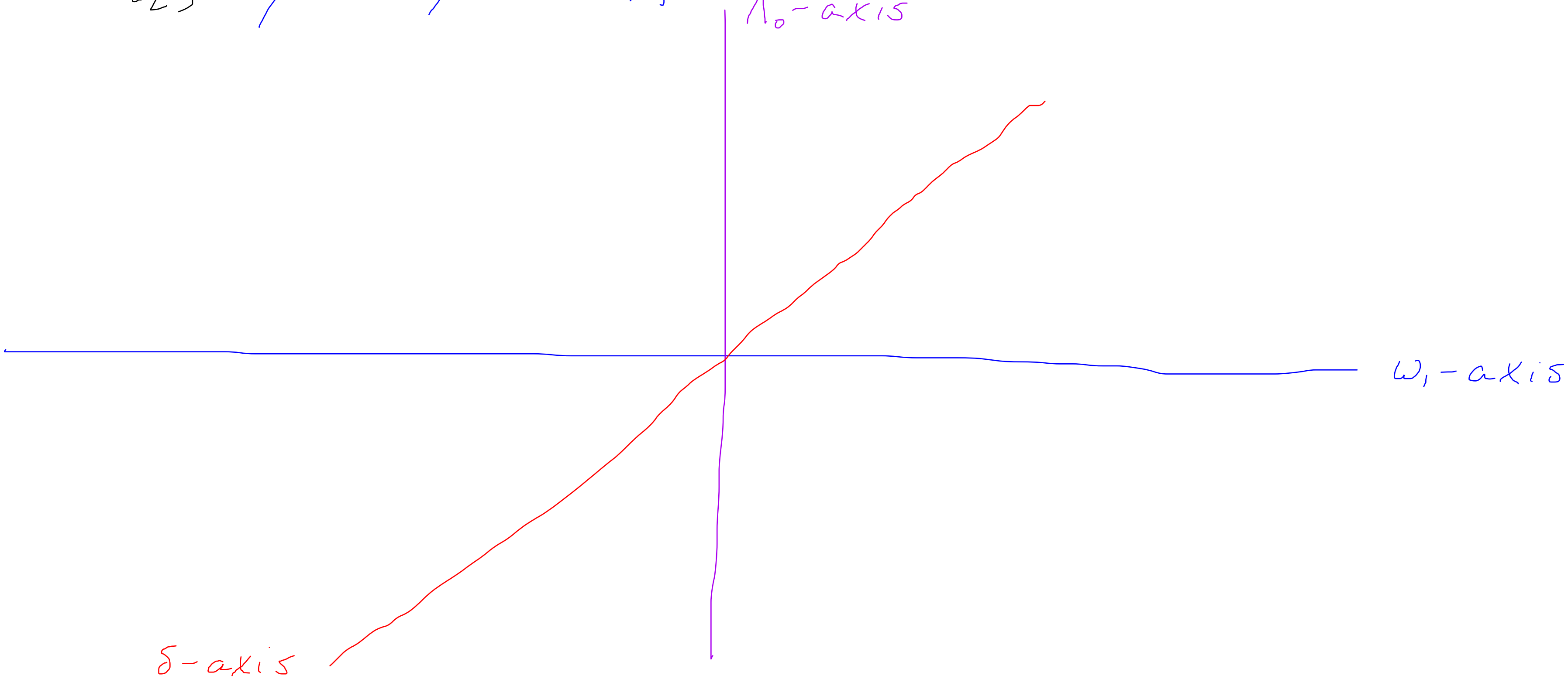

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Then

$$t_\beta \cdot \lambda = \left(\begin{array}{c|c|c} 1 & -\beta & -\frac{1}{2}(\beta|\beta) \\ \hline 0 & 1 & \beta \\ \hline 0 & 0 & 1 \end{array}\right) \begin{pmatrix} a \\ \hline \gamma \\ \hline m \end{pmatrix}$$

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Formula (6.5.2) from Kac, Infinite dim. Lie algebras

The <u>affine</u> <u>Weyl</u> <u>group</u> is

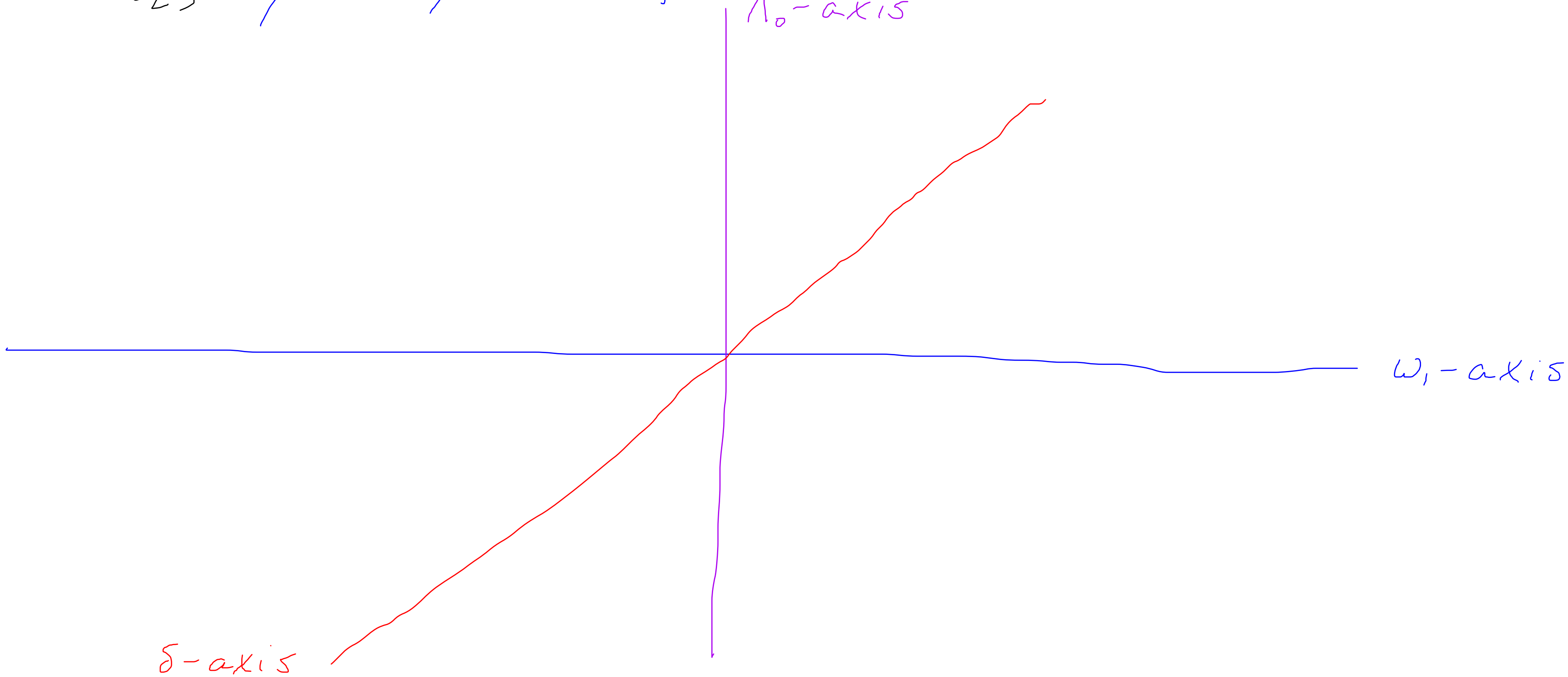
$$W = \{ w t_\beta \mid \beta \in \mathring{h}_{\mathbb{Z}},\ w \in W_0 \}$$

The <u>affine Weyl group</u> $W = \{ w t_\beta \mid \beta \in \mathring{\mathfrak{h}}_{\mathbb{Z}},\ w \in W_0 \}$

acts on

$$\mathfrak{h}^* = \mathbb{C}\delta \oplus \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 = \left\{ \begin{pmatrix} a \\ \hline \vdots \\ \gamma \\ \vdots \\ \hline \ell \end{pmatrix} \;\middle|\; a, \ell \in \mathbb{C},\ \gamma \in \mathring{\mathfrak{h}}^* \right\}$$

For sl_2, $\mathring{\mathfrak{h}}^* = \mathrm{span}\{\omega_1\}$



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$\mathring{\mathfrak{h}}^*_{\mathbb{Z}} = Hom(T, \mathbb{C})$ $\qquad$ $\mathring{\mathfrak{h}}_{\mathbb{Z}} = Hom(\mathbb{C}^\times, T)$ $\qquad$ $W_0 = N_G(T)/T$

G is a complex reductive algebraic group

$\cup|$

T is a maximal torus

For $G = SL_2$:

$\mathring{\mathfrak{h}}^*_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\omega_1\}$

$-13\omega_1 \quad -12\omega_1 \quad -11\omega_1 \quad -10\omega_1 \quad -9\omega_1 \quad -8\omega_1 \quad -7\omega_1 \quad -6\omega_1 \quad -6\omega_1 \quad -5\omega_1 \quad -4\omega_1 \quad -3\omega_1 \quad -2\omega_1 \quad -\omega_1 \quad 0 \quad \omega_1 \quad 2\omega_1 \quad 3\omega_1 \quad 4\omega_1 \quad 5\omega_1 \quad 6\omega_1 \quad 7\omega_1 \quad 8\omega_1 \quad 9\omega_1 \quad 10\omega_1 \quad 11\omega_1 \quad 12\omega_1 \quad 13\omega_1 \quad 14\omega_1 \quad 15\omega_1$

$\mathring{\mathfrak{h}}^*_{\mathbb{Z}} = \mathrm{Hom}(T, \mathbb{C})$ $\qquad$ $\mathring{\mathfrak{h}}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^\times, T)$ $\qquad$ $W_0 = N_G(T)/T$

For $G = SL_2$:

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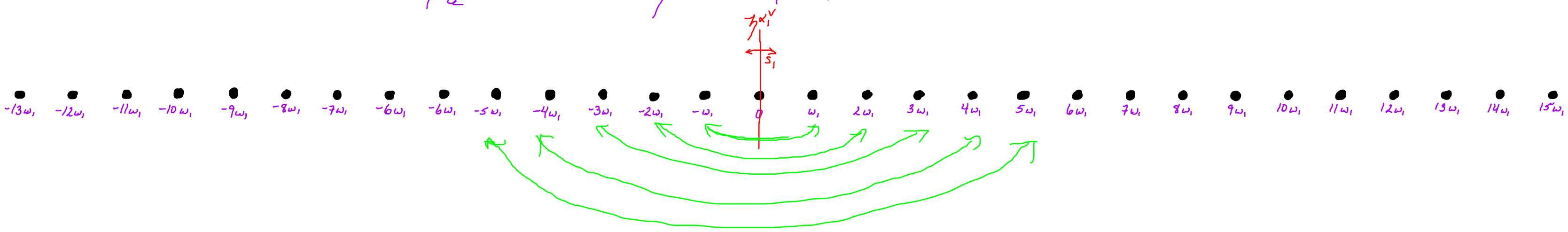
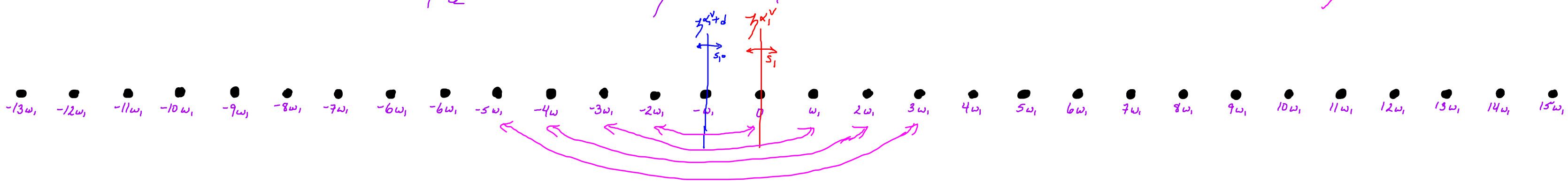
W_0 acts on $\mathring{\mathfrak{h}}^*_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\omega_1\}$



W_0 acts on $\mathring{\mathfrak{h}}^*_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\omega_1\}$ (dot action $W_0 \circ$)



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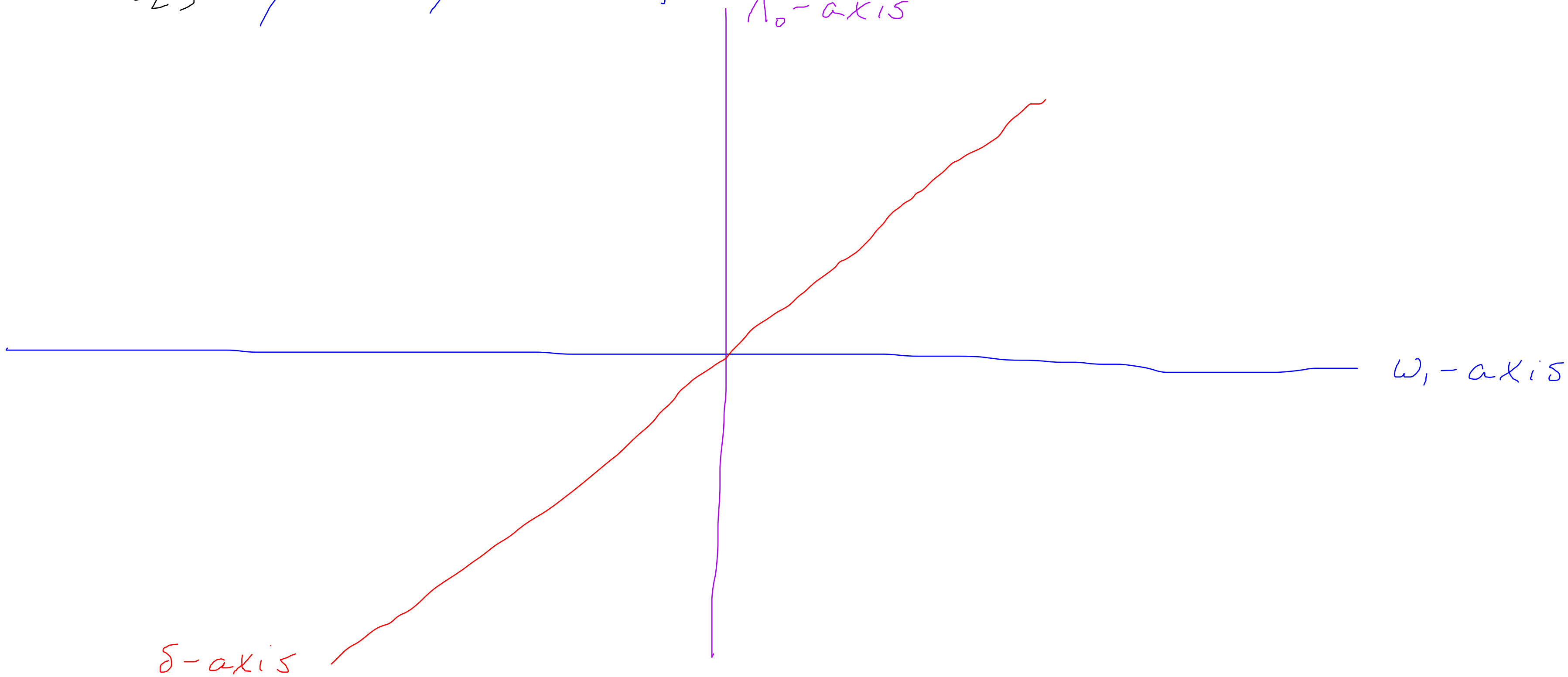

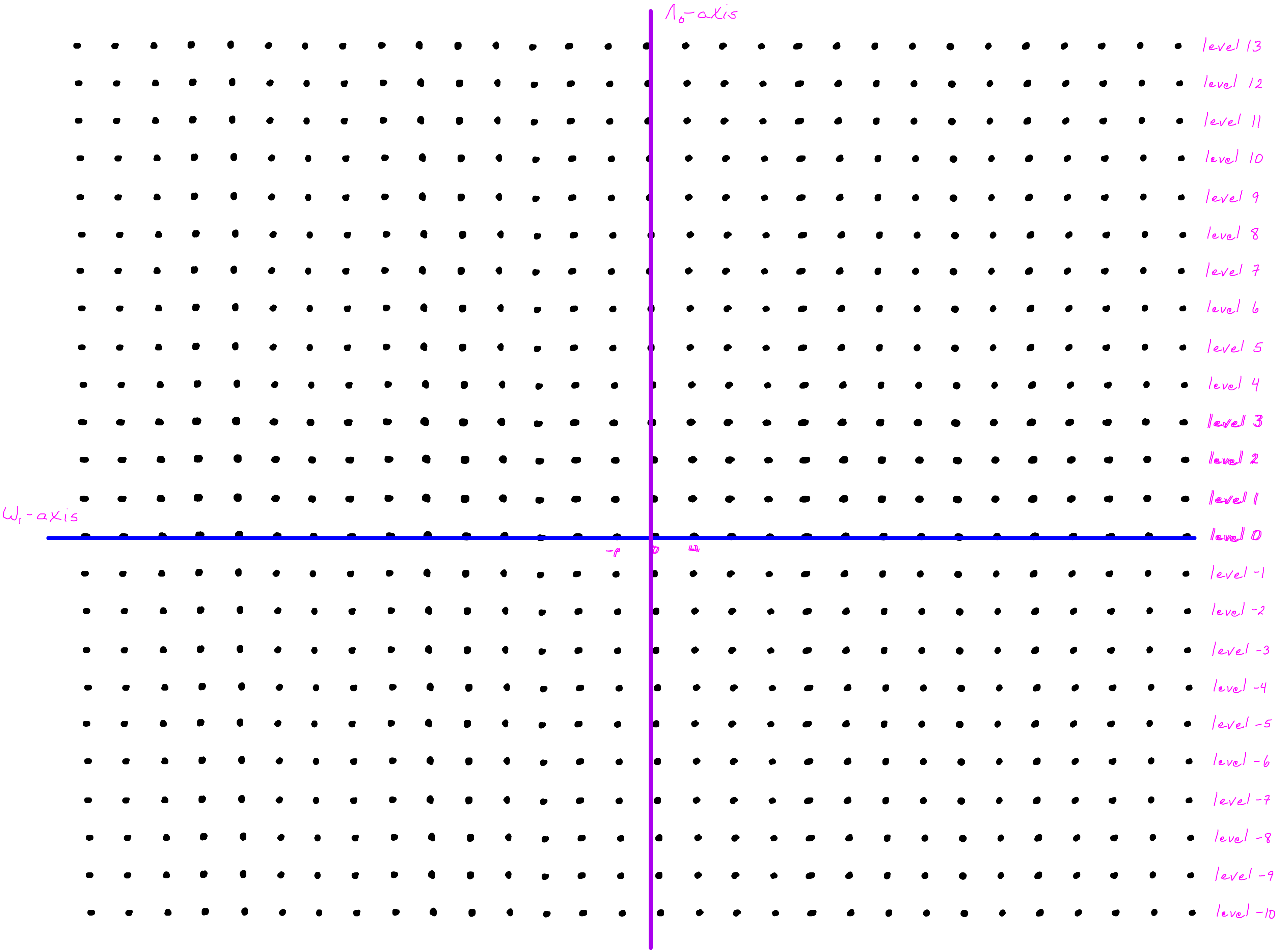

Λ_0-axis
ω_1-axis
level 13
level 12
level 11
level 10
level 9
level 8
level 7
level 6
level 5
level 4
level 3
level 2
level 1
level 0
level -1
level -2
level -3
level -4
level -5
level -6
level -7
level -8
level -9
level -10
$-\rho$
0
ω_1

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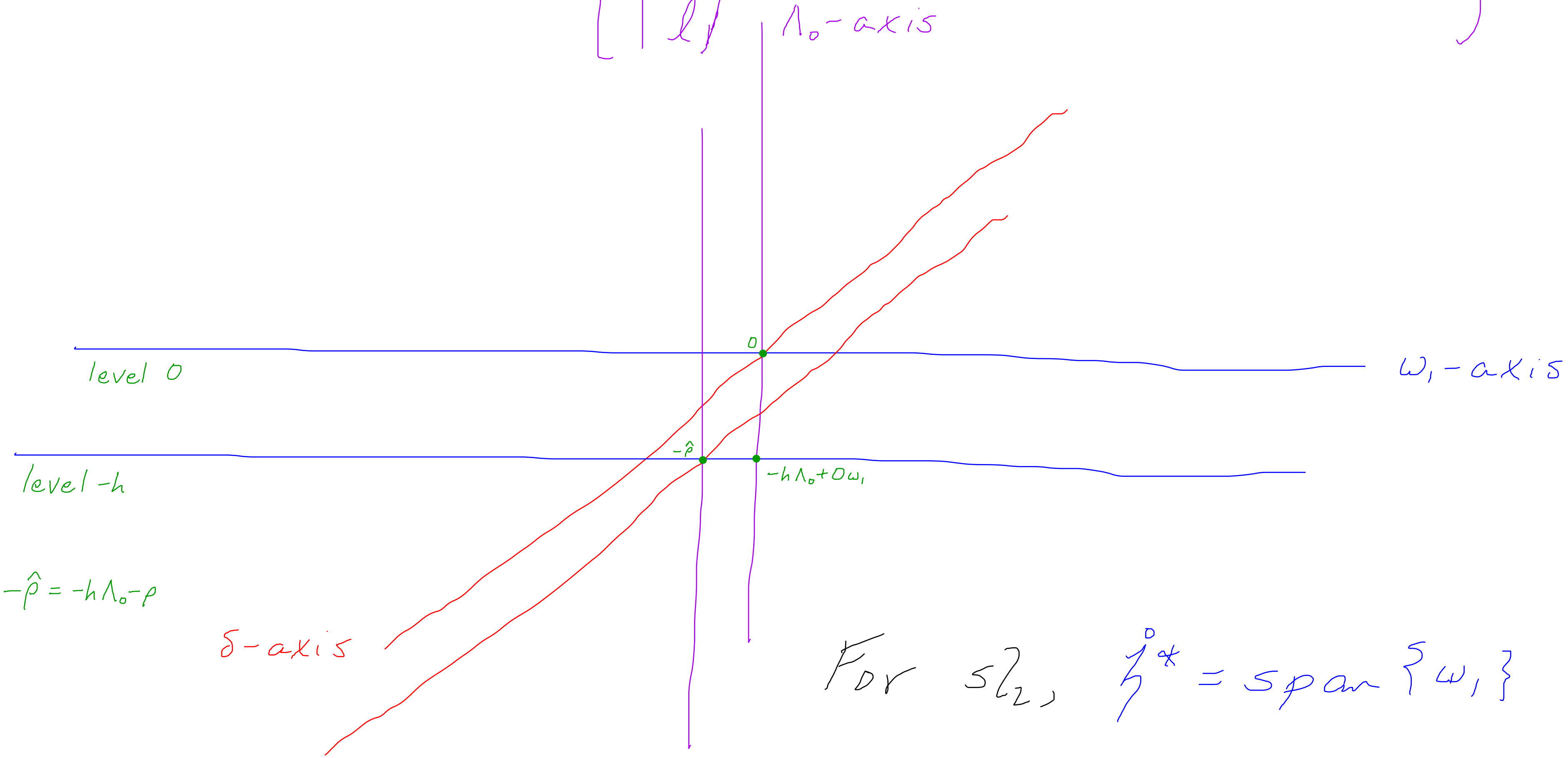
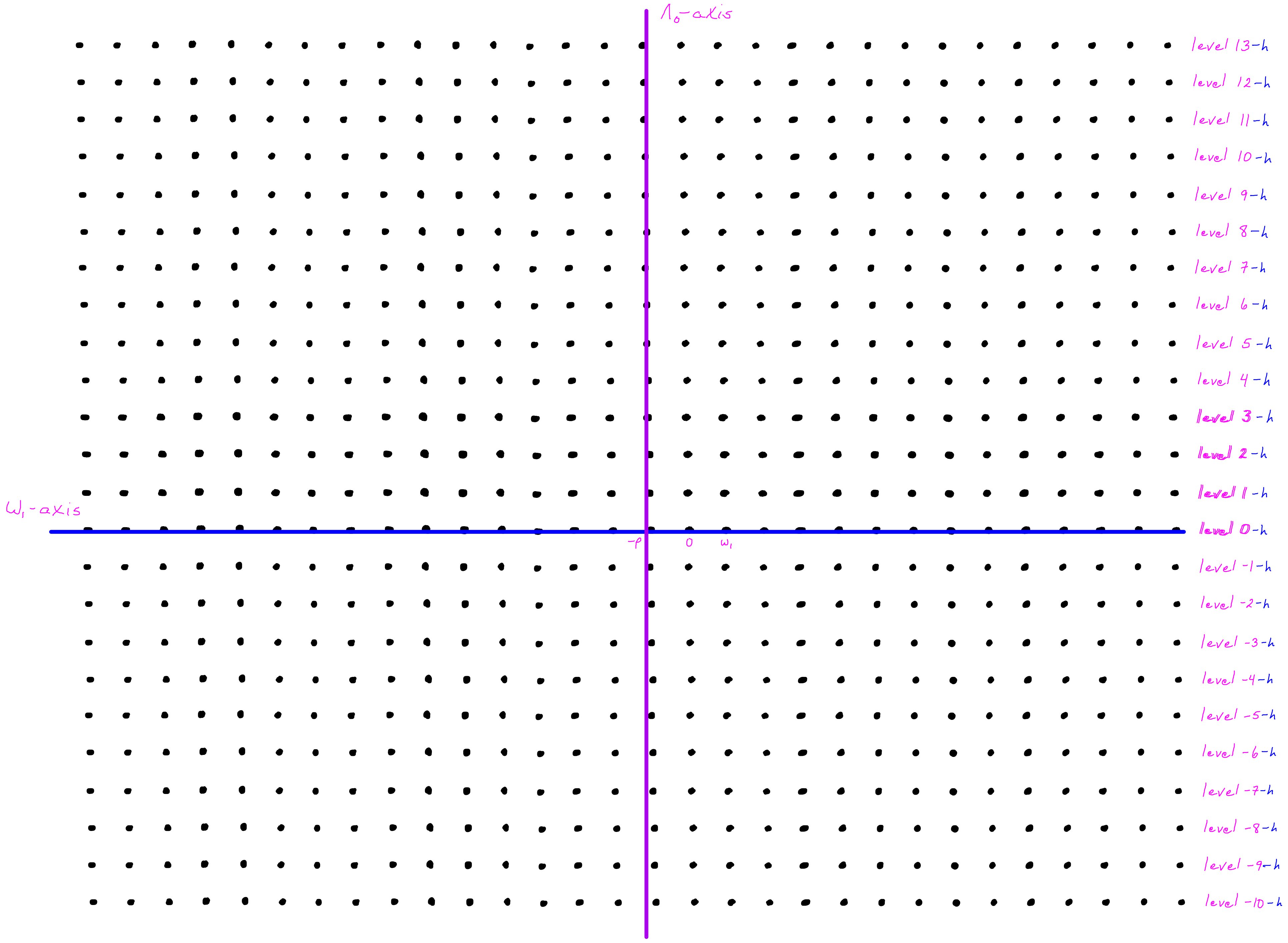
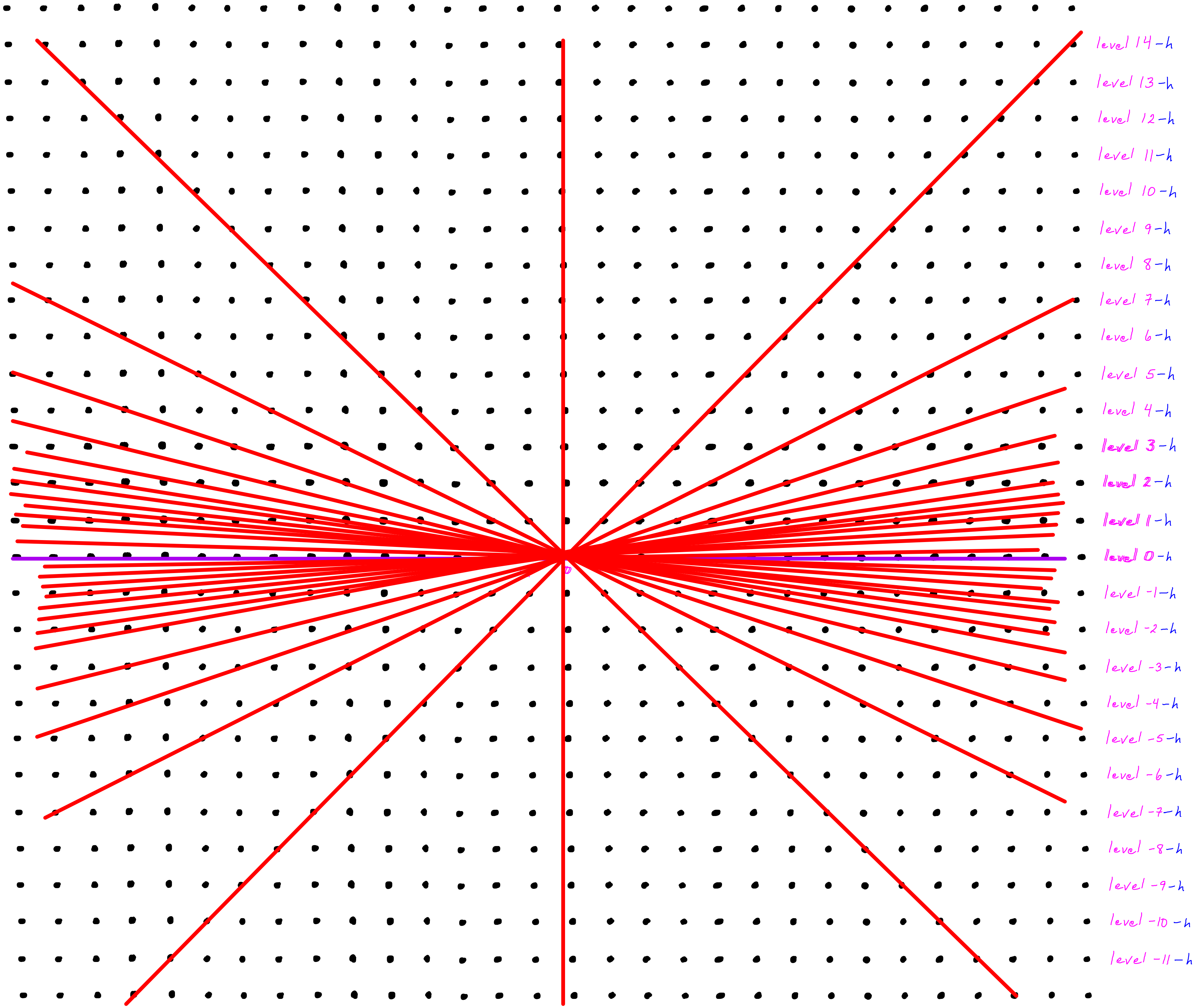


For $\mathfrak{sl}_2$, $\mathring{\mathfrak{h}}^* = \mathrm{span}\{\omega_1\}$



Λ_0-axis
ω_1-axis
$-\rho$
0
ω_1
level 13-h
level 12-h
level 11-h
level 10-h
level 9-h
level 8-h
level 7-h
level 6-h
level 5-h
level 4-h
level 3-h
level 2-h
level 1-h
level 0-h
level -1-h
level -2-h
level -3-h
level -4-h
level -5-h
level -6-h
level -7-h
level -8-h
level -9-h
level -10-h


level 14-h
level 13-h
level 12-h
level 11-h
level 10-h
level 9-h
level 8-h
level 7-h
level 6-h
level 5-h
level 4-h
level 3-h
level 2-h
level 1-h
level 0-h
level -1-h
level -2-h
level -3-h
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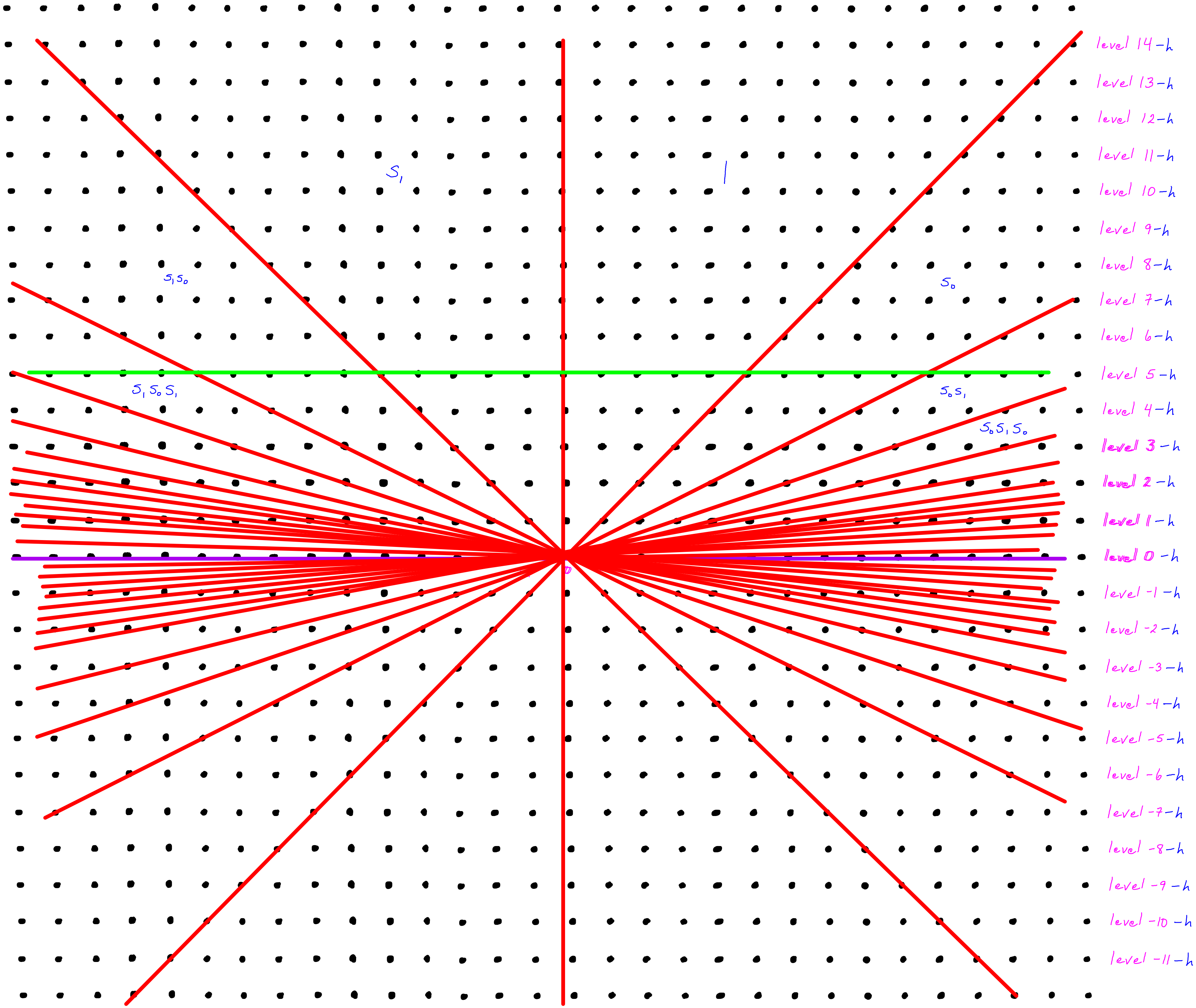
level -5-h
level -6-h
level -7-h
level -8-h
level -9-h
level -10-h
level -11-h



S₁
1
S₁S₀
S₀
S₁S₀S₁
S₀S₁
S₀S₁S₀
0
level 14-h
level 13-h
level 12-h
level 11-h
level 10-h
level 9-h
level 8-h
level 7-h
level 6-h
level 5-h
level 4-h
level 3-h
level 2-h
level 1-h
level 0-h
level -1-h
level -2-h
level -3-h
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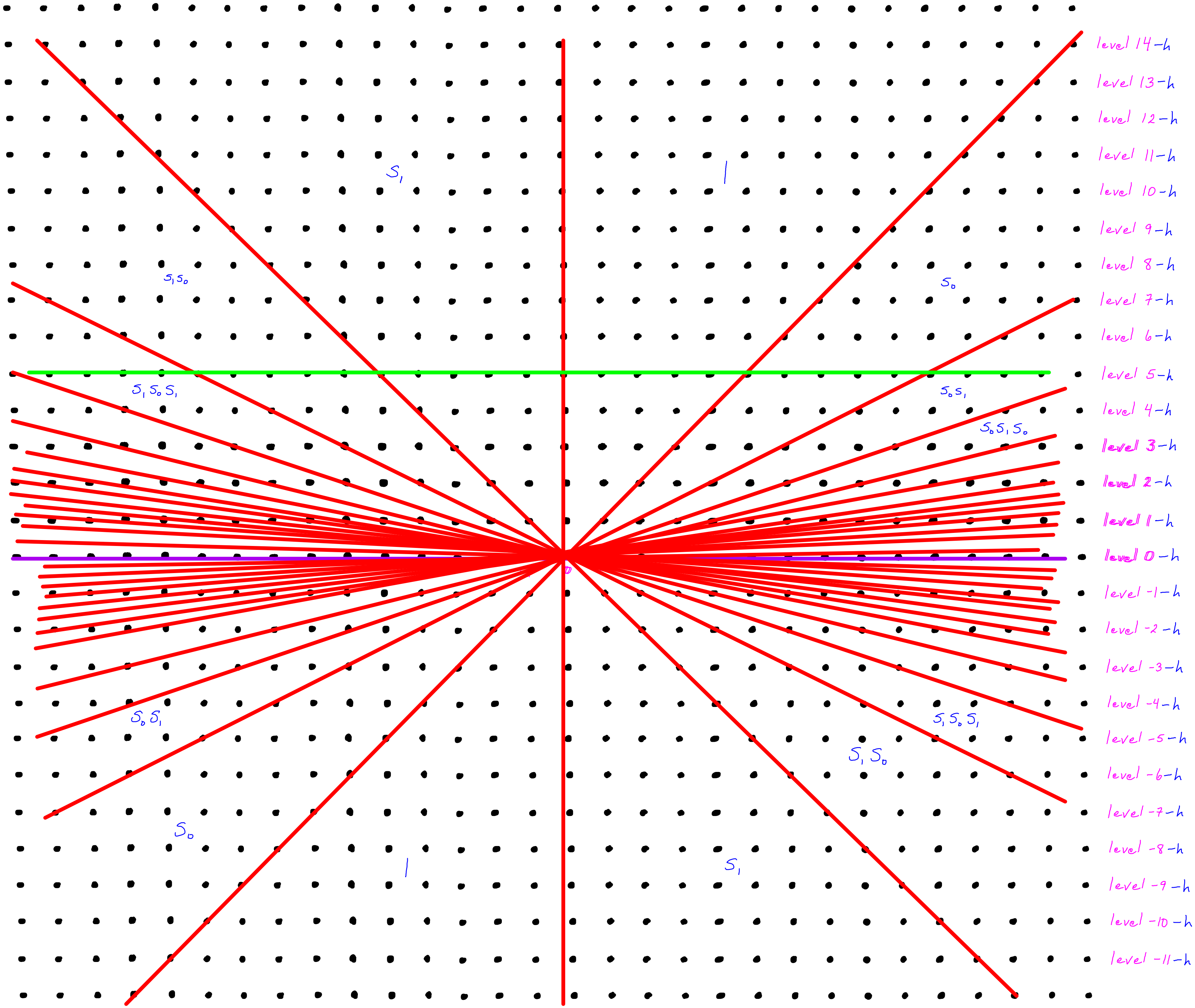
level -8-h
level -9-h
level -10-h
level -11-h



level 14-h
level 13-h
level 12-h
level 11-h
level 10-h
level 9-h
level 8-h
level 7-h
level 6-h
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level 3-h
level 2-h
level 1-h
level 0-h
level -1-h
level -2-h
level -3-h
level -4-h
level -5-h
level -6-h
level -7-h
level -8-h
level -9-h
level -10-h
level -11-h
S_1
1
S_1S_0
S_0
$S_1S_0S_1$
S_0S_1
$S_0S_1S_0$
S_0S_1
$S_1S_0S_1$
S_1S_0
S_0
1
S_1

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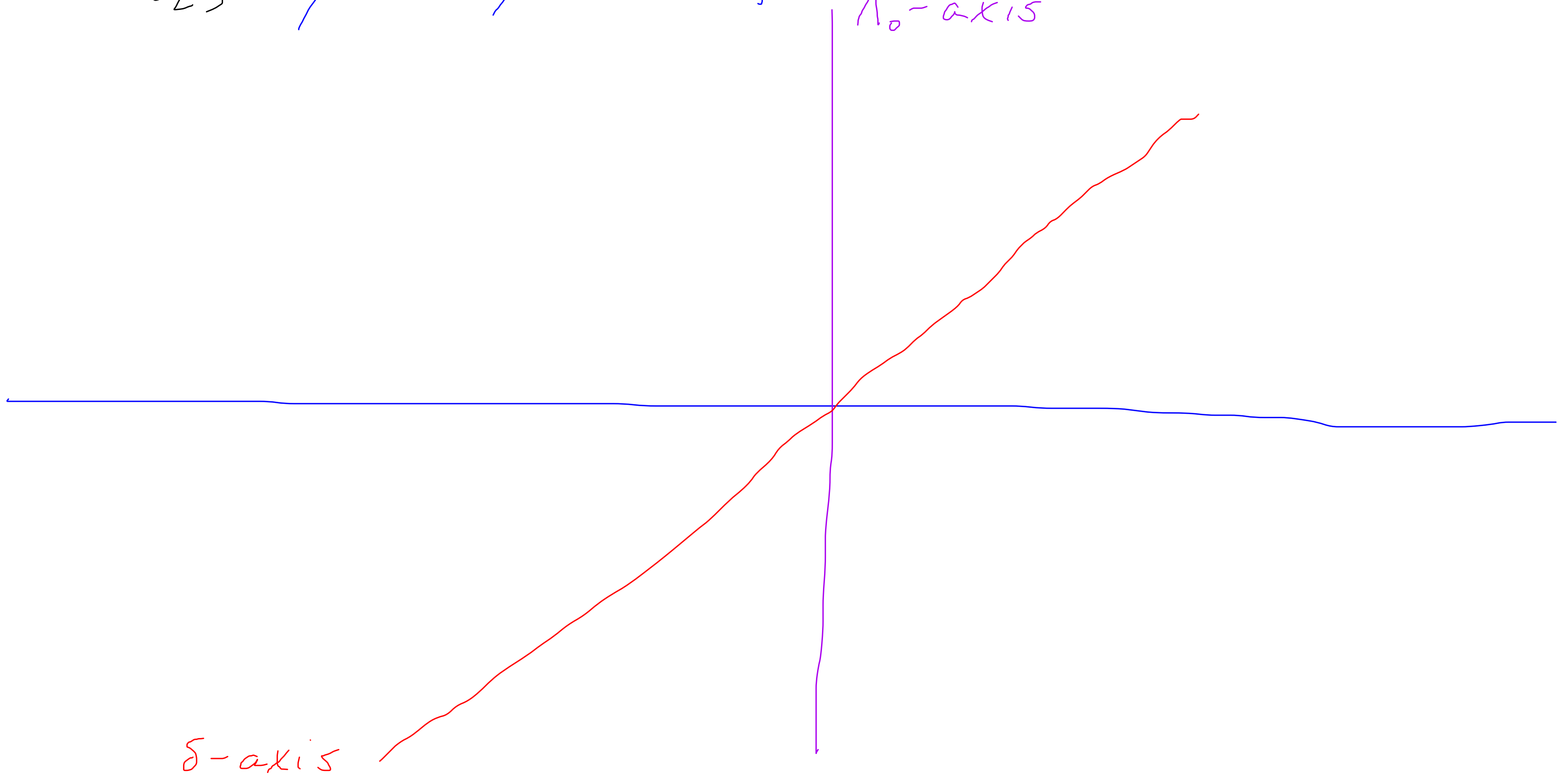

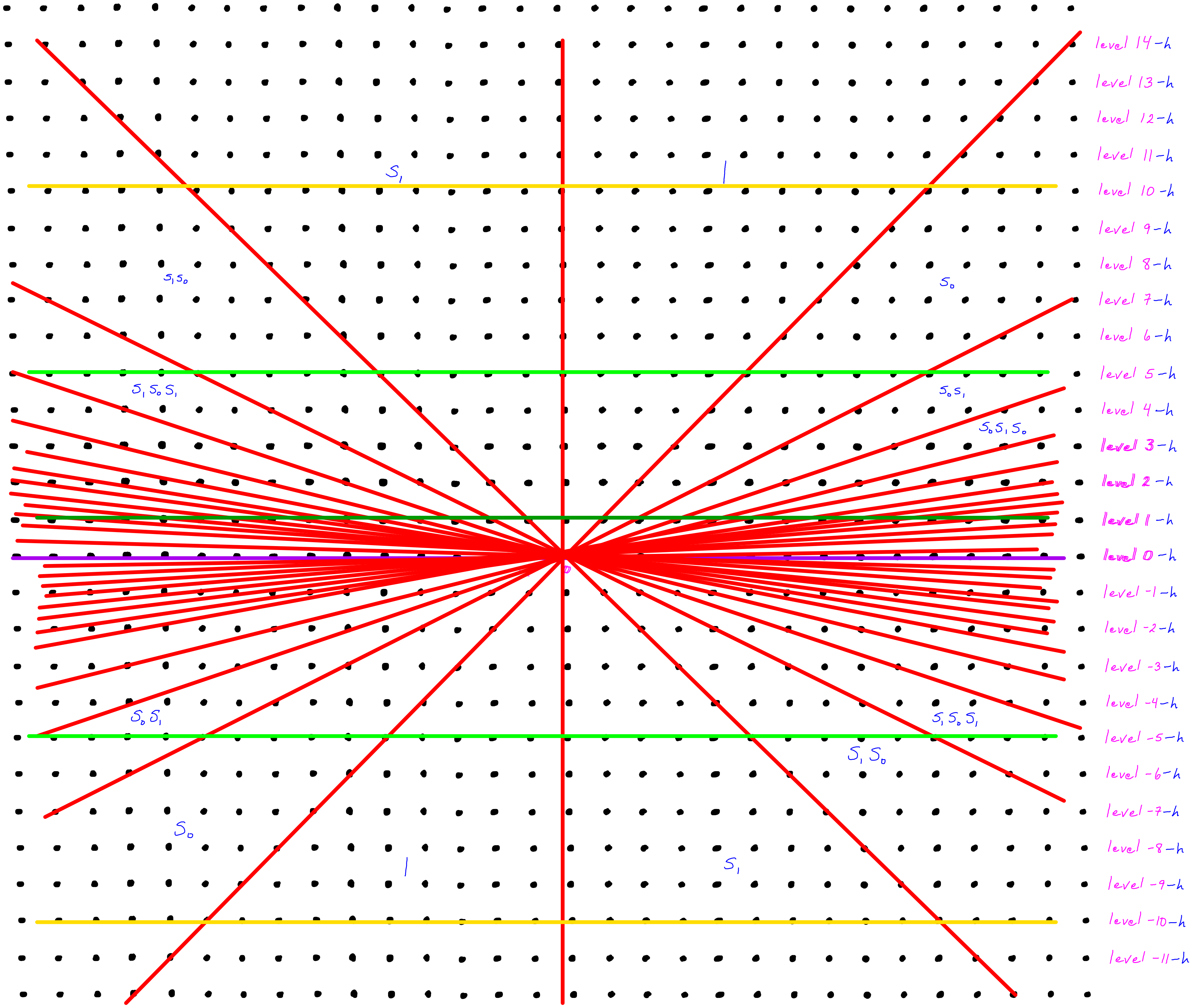

level 14-h
level 13-h
level 12-h
level 11-h
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level 9-h
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level 5-h
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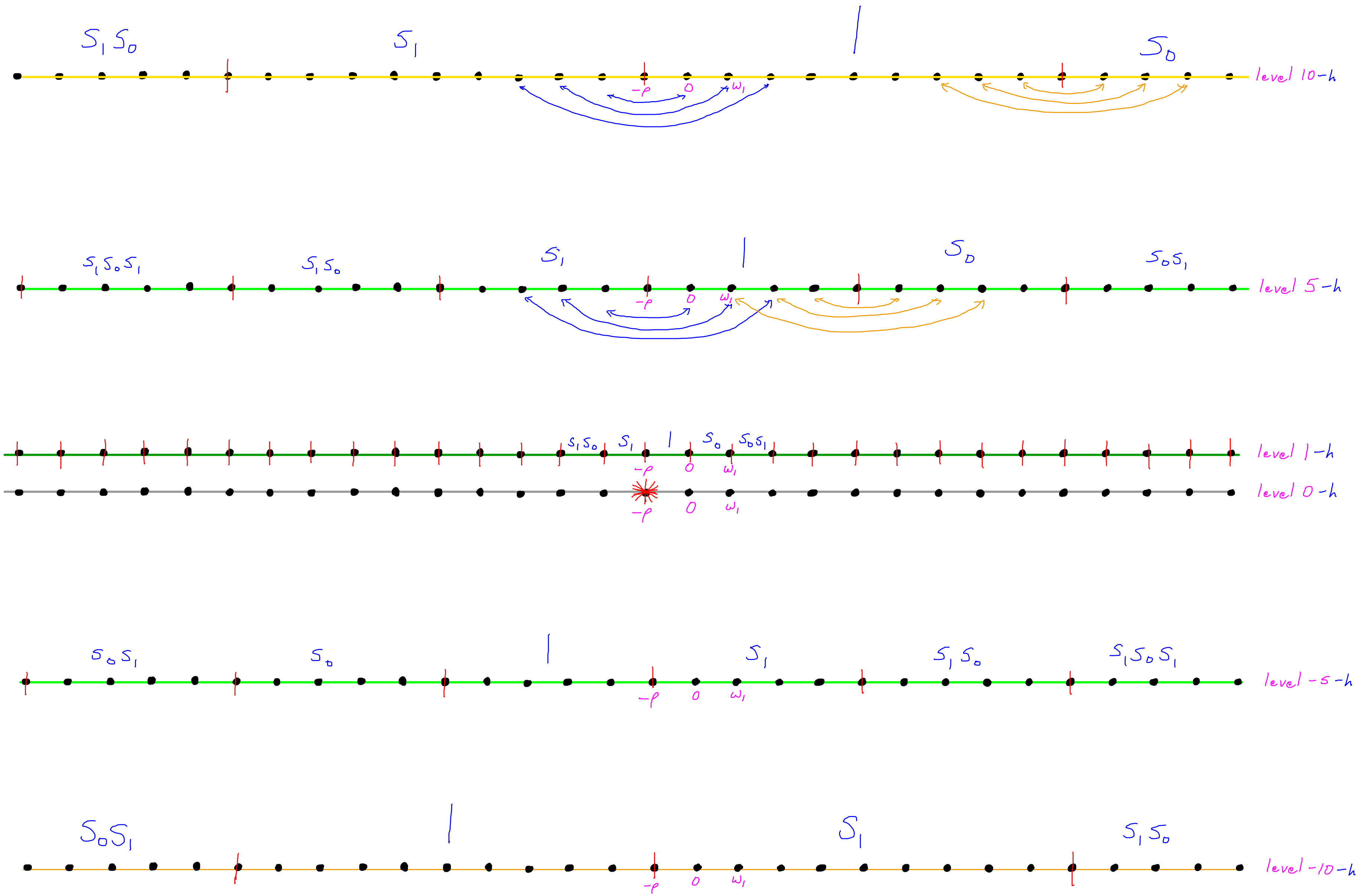
level 0-h
level -1-h
level -2-h
level -3-h
level -4-h
level -5-h
level -6-h
level -7-h
level -8-h
level -9-h
level -10-h
level -11-h
S1
1
S1 S0
S0
S1 S0 S1
S0 S1
S0 S1 S0
S0 S1
S1 S0 S1
S1 S0
S0
1
S1



$s_1 s_0$
s_1
1
s_0
$-\rho$
0
ω_1
level 10-h
$s_1 s_0 s_1$
$s_1 s_0$
s_1
1
s_0
$s_0 s_1$
$-\rho$
0
ω_1
level 5-h
$s_1 s_0$
s_1
1
s_0
$s_0 s_1$
$-\rho$
0
ω_1
level 1-h
$-\rho$
0
ω_1
level 0-h
$s_0 s_1$
s_0
1
s_1
$s_1 s_0$
$s_1 s_0 s_1$
$-\rho$
0
ω_1
level -5-h
$s_0 s_1$
1
s_1
$s_1 s_0$
$-\rho$
0
ω_1
level -10-h

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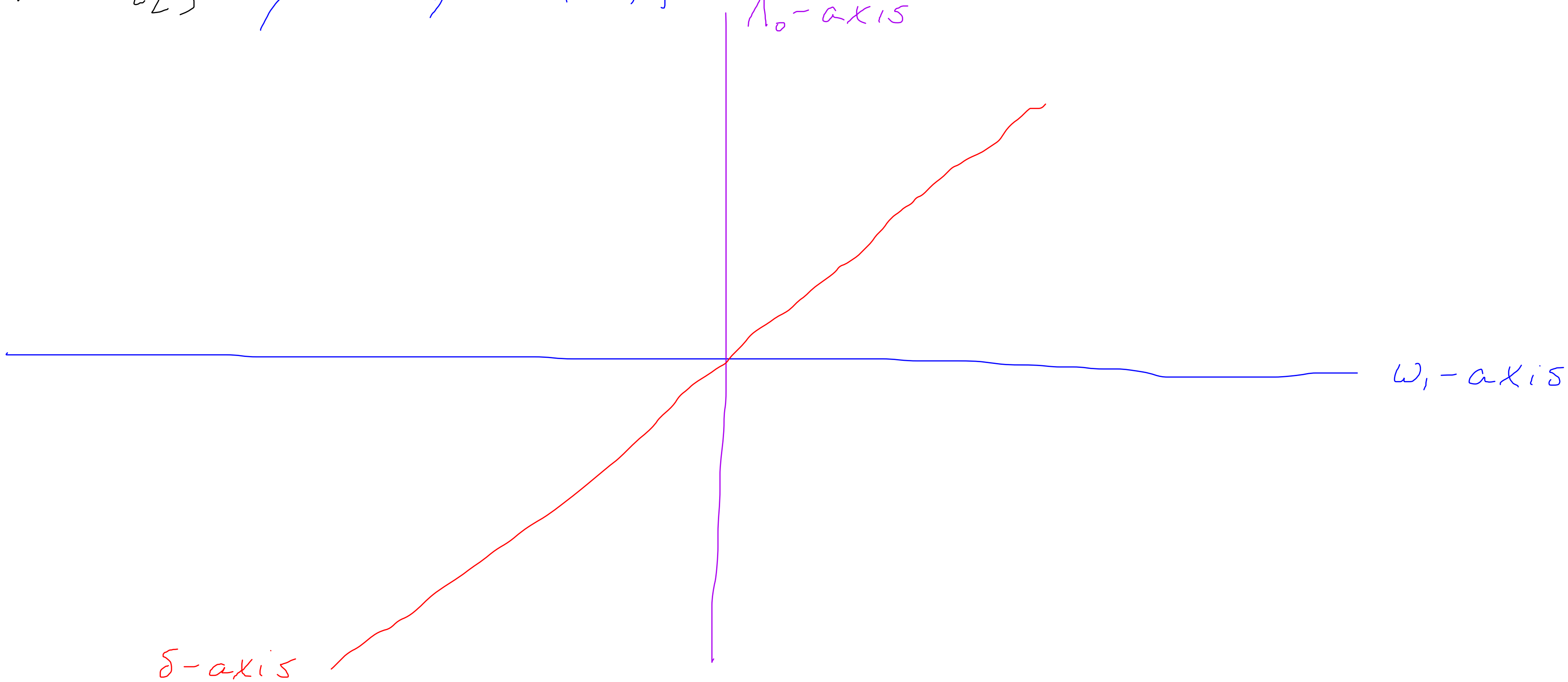
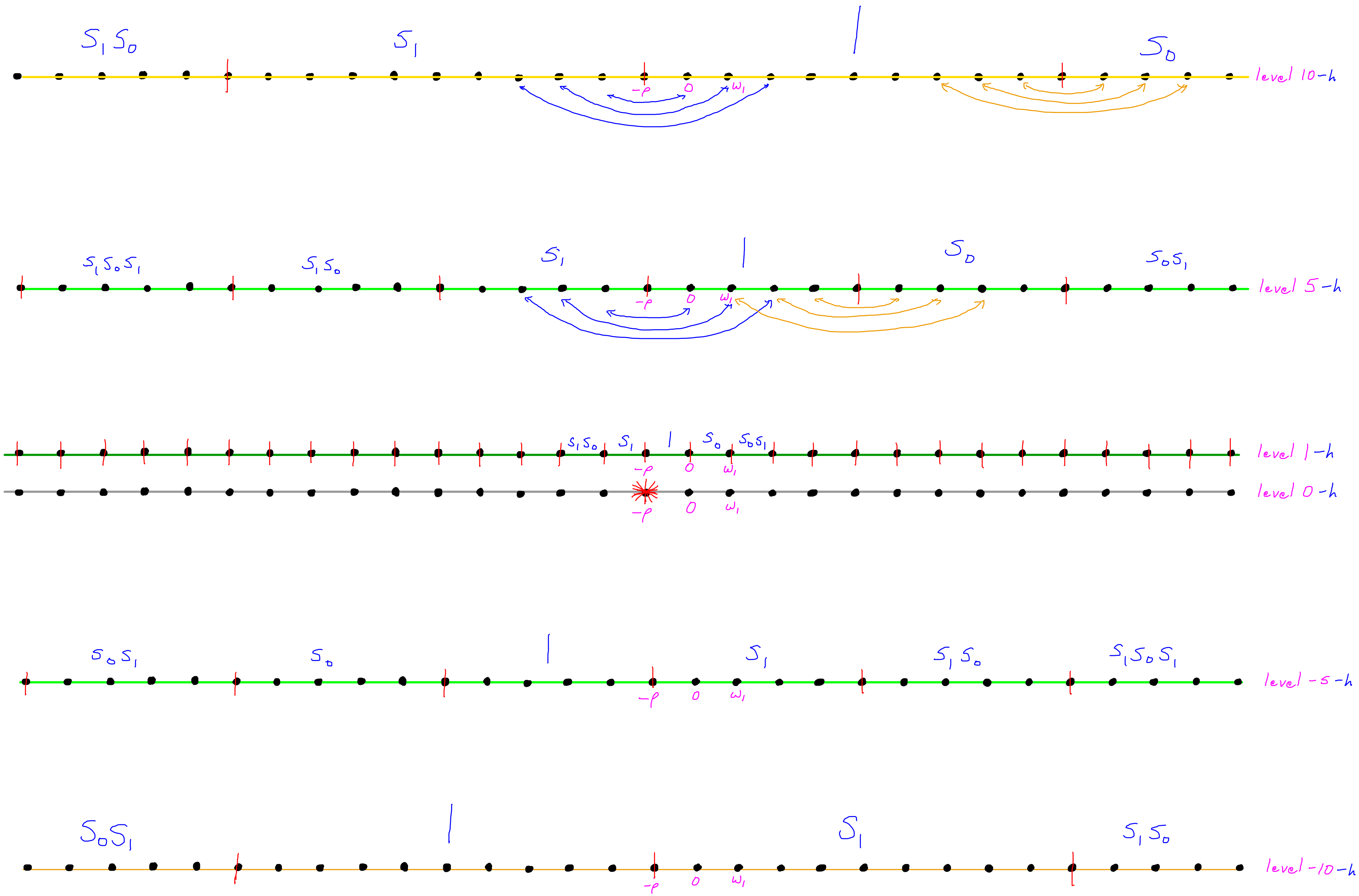
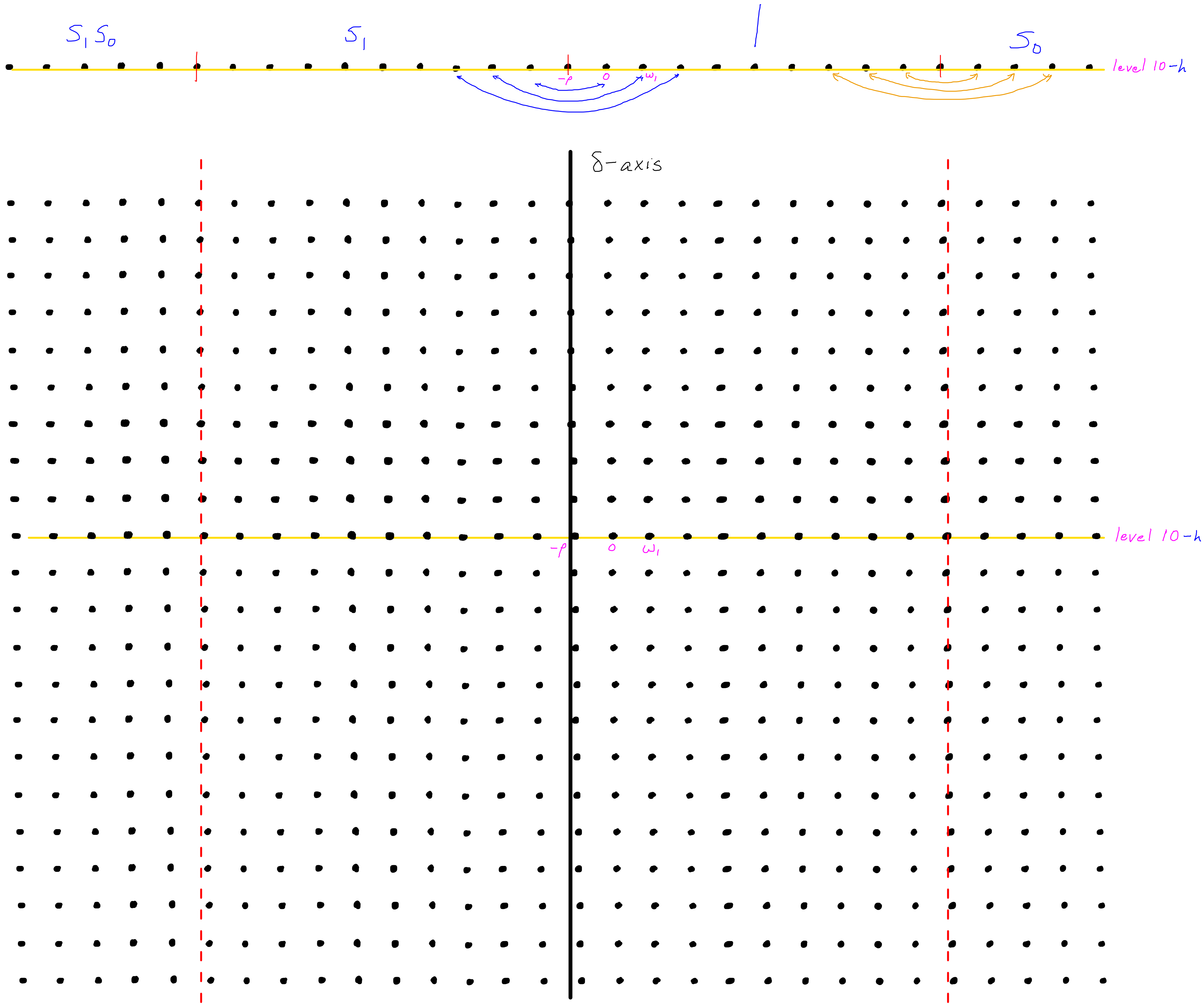
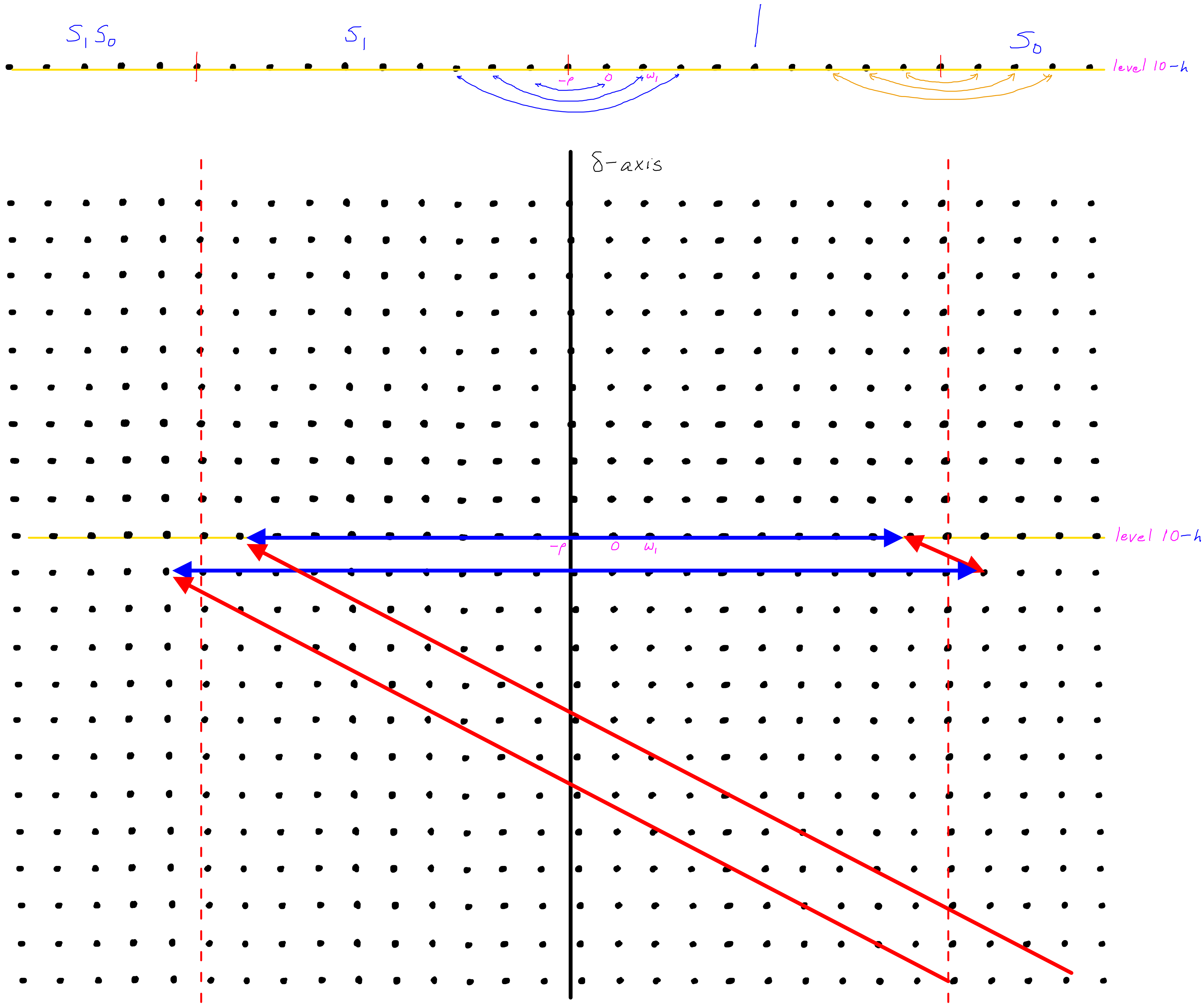




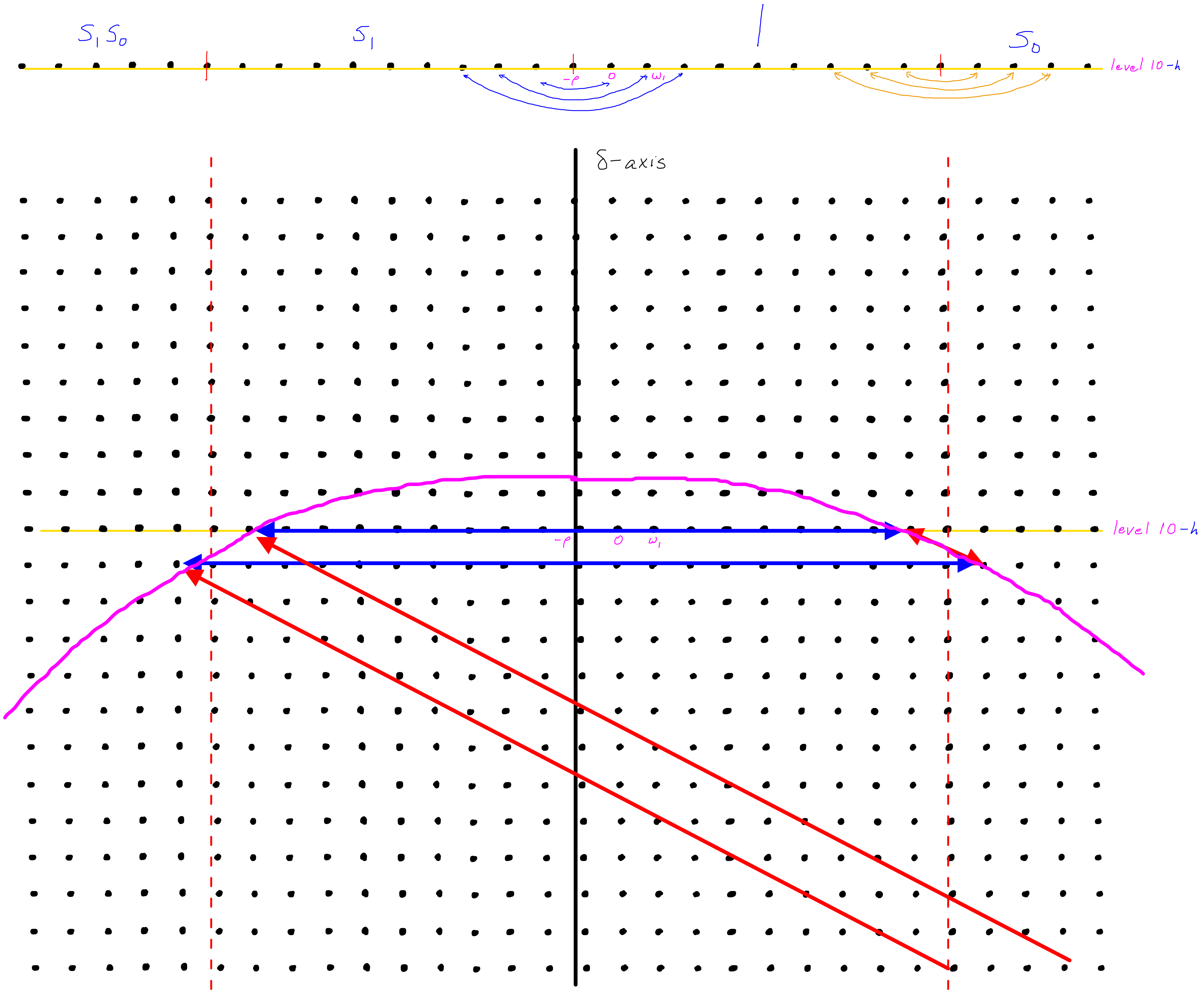
$s_1 s_0$
s_1
1
s_0
$-\rho$
0
ω_1
level 10-h
$s_1 s_0 s_1$
$s_1 s_0$
s_1
1
s_0
$s_0 s_1$
$-\rho$
0
ω_1
level 5-h
$s_1 s_0$
s_1
1
s_0
$s_0 s_1$
$-\rho$
0
ω_1
level 1-h
$-\rho$
0
ω_1
level 0-h
$s_0 s_1$
s_0
1
s_1
$s_1 s_0$
$s_1 s_0 s_1$
$-\rho$
0
ω_1
level -5-h
$s_0 s_1$
1
s_1
$s_1 s_0$
$-\rho$
0
ω_1
level -10-h



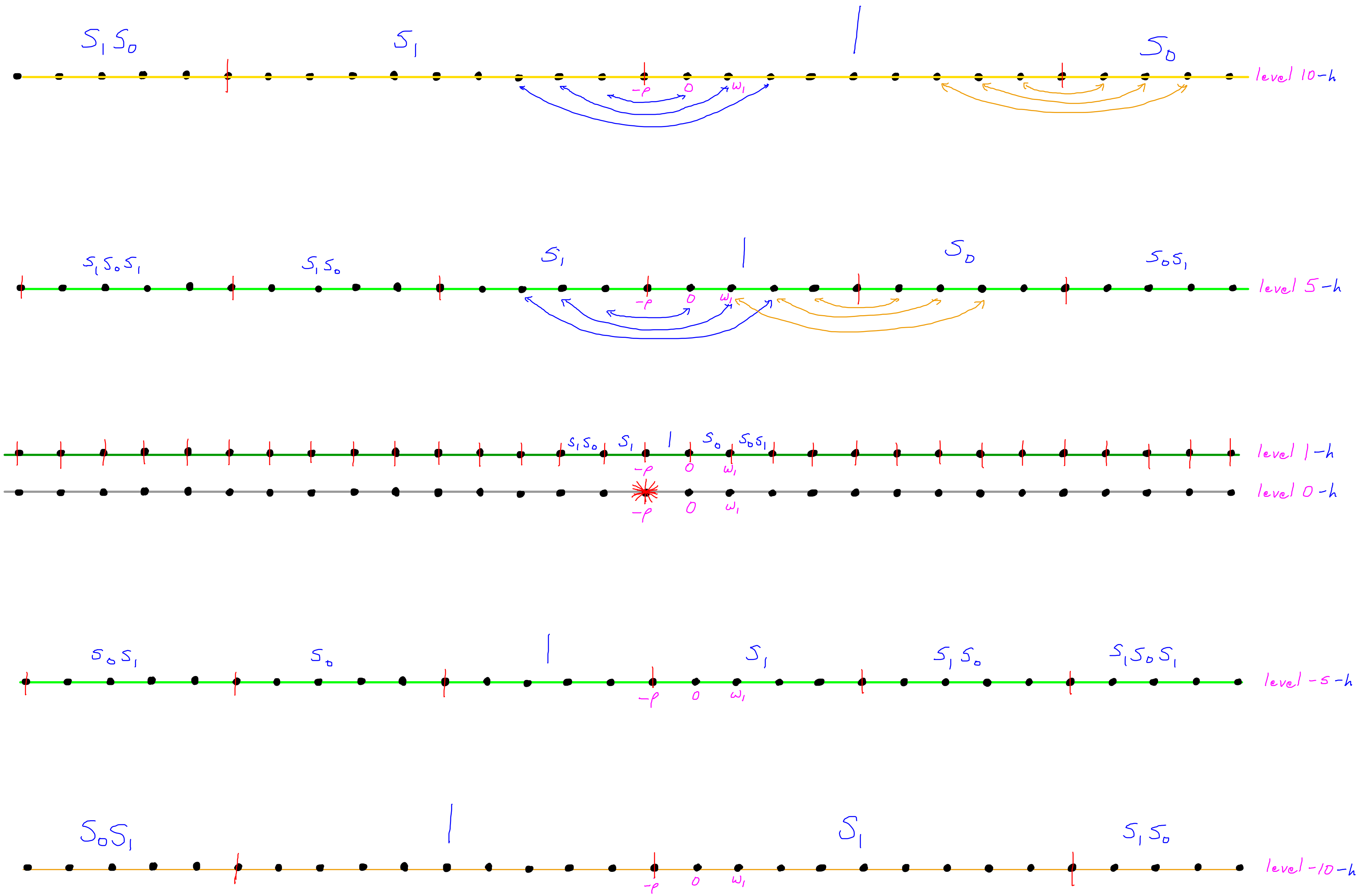
$S_1 S_0$
S_1
S_0
level 10-h
$-\rho$
0
ω_1
δ-axis
level 10-h
$-\rho$
0
ω_1



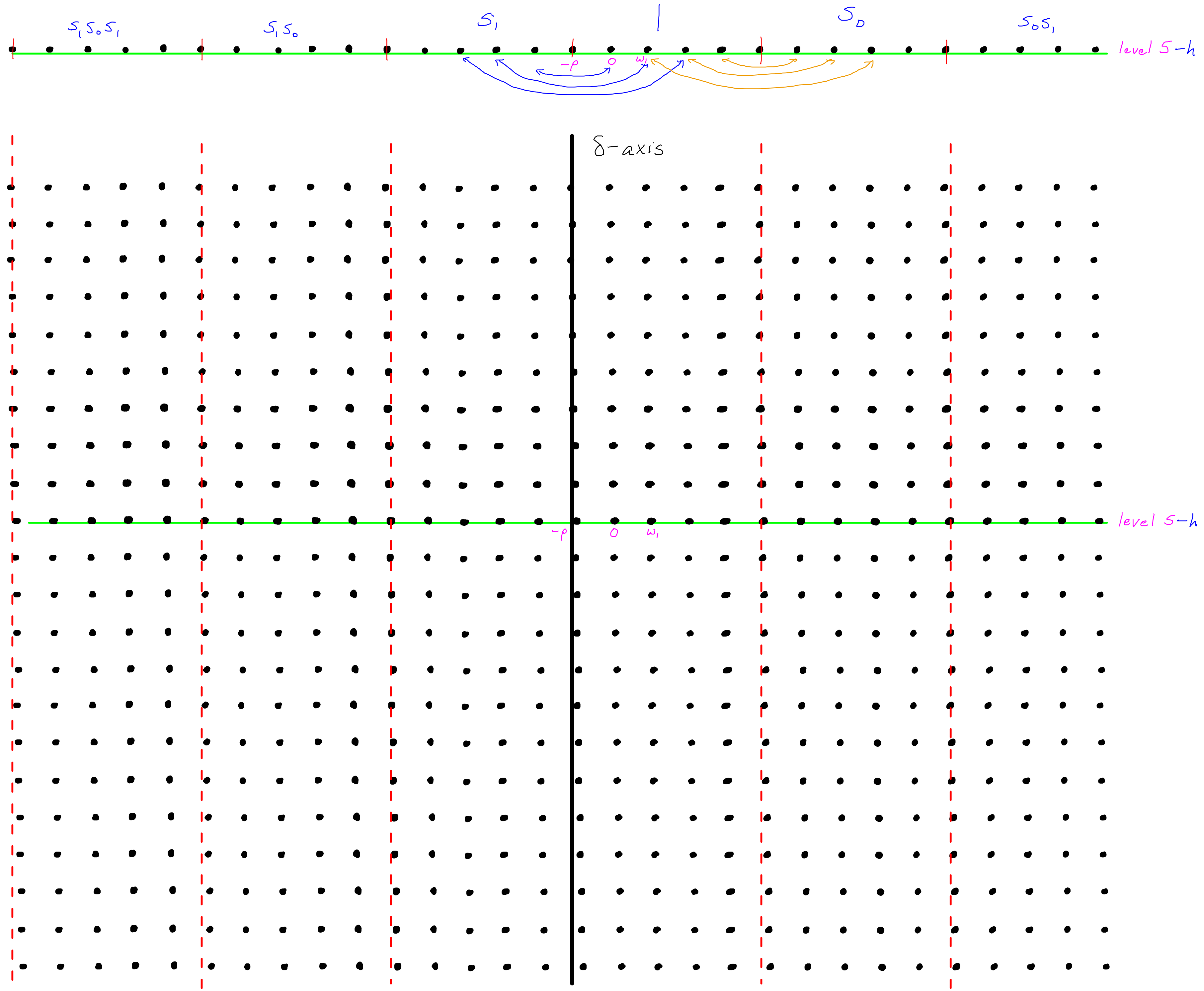
$S_1 S_0$
S_1
S_0
level 10−h
$-\rho$
0
ω_1
δ-axis
level 10−h
$-\rho$
0
ω_1



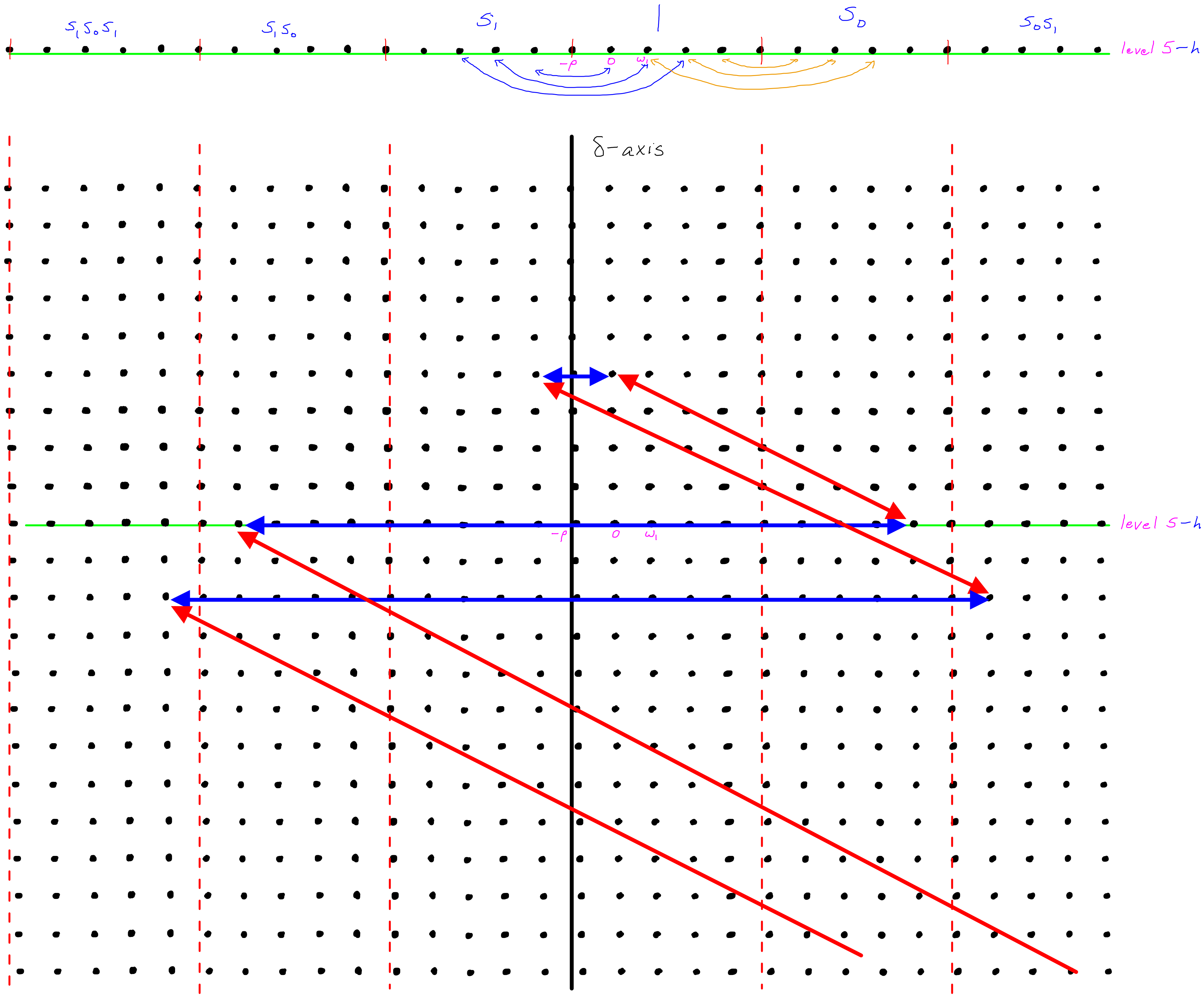
$S_1 S_0$
S_1
S_0
level 10-h
$-\rho$
0
ω_1
δ-axis
level 10-h
$-\rho$
0
ω_1



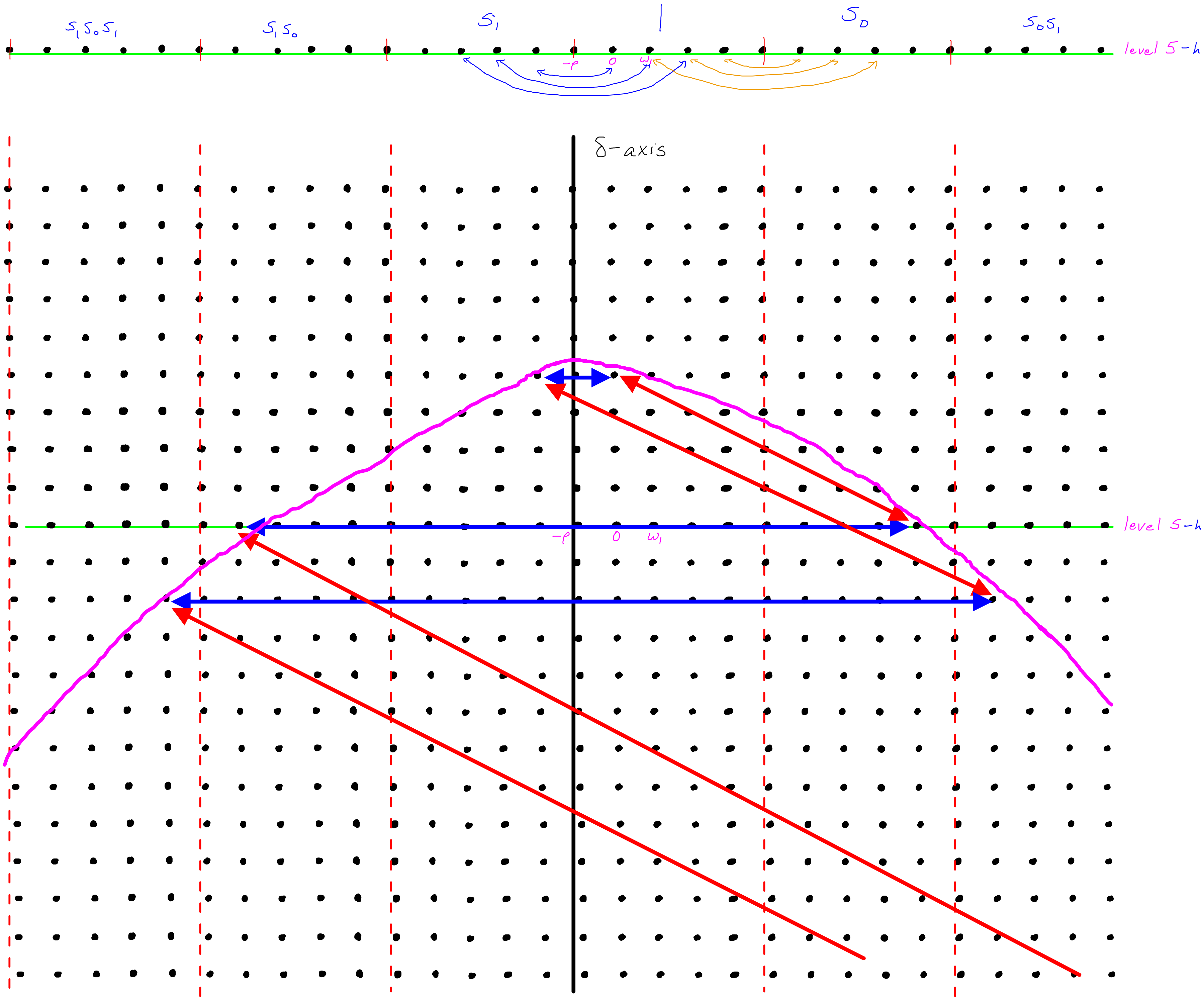
S1 S0
S1
1
S0
-ρ
0
ω1
level 10-h
S1 S0 S1
S1 S0
S1
1
S0
S0 S1
-ρ
0
ω1
level 5-h
S1 S0
S1
1
S0
S0 S1
-ρ
0
ω1
level 1-h
-ρ
0
ω1
level 0-h
S0 S1
S0
1
S1
S1 S0
S1 S0 S1
-ρ
0
ω1
level -5-h
S0 S1
1
S1
S1 S0
-ρ
0
ω1
level -10-h



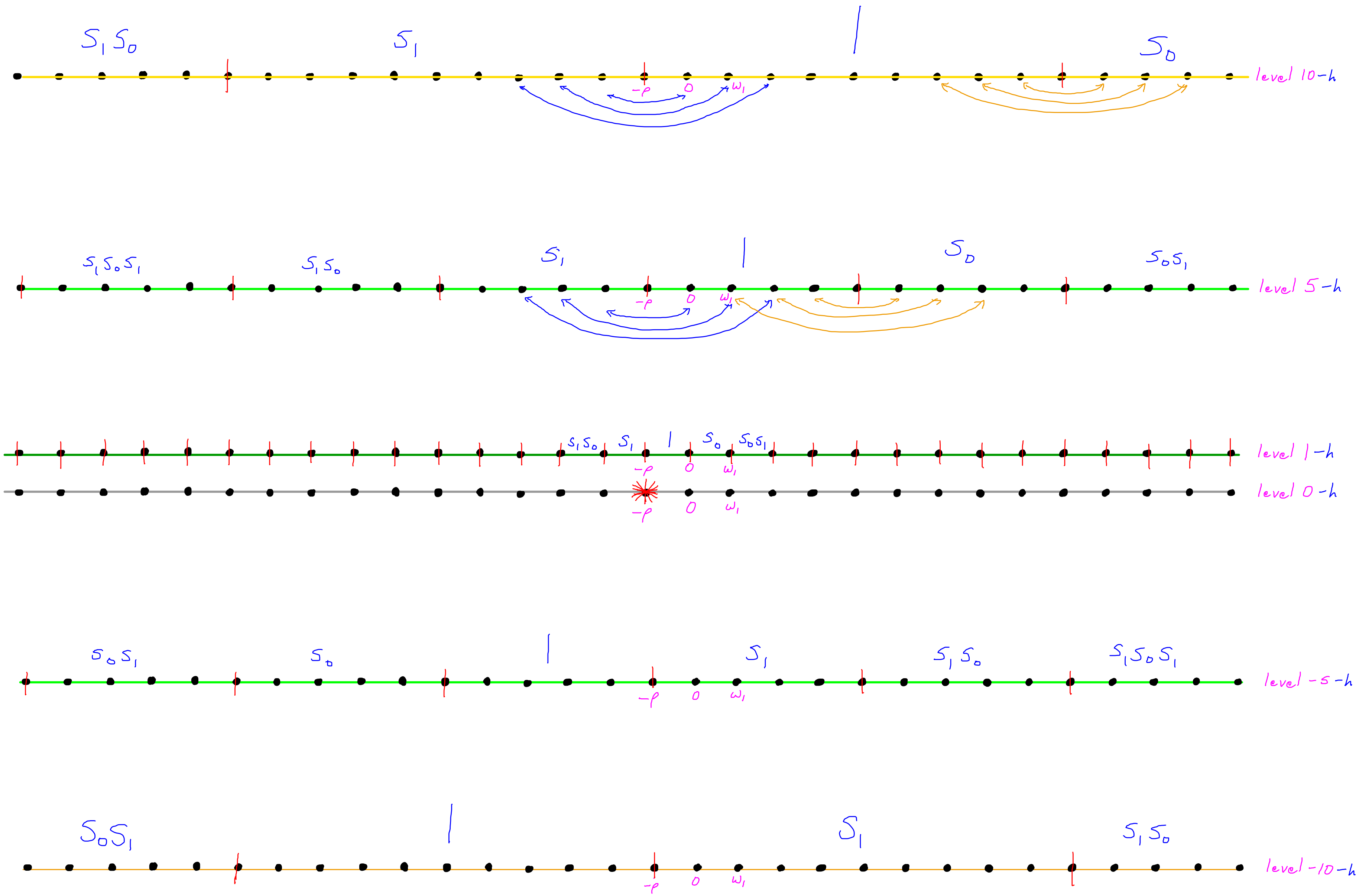
$s_1 s_0 s_1$
$s_1 s_0$
s_1
1
s_0
$s_0 s_1$
level $5-h$
$-\rho$
0
ω_1
δ-axis
level $5-h$
$-\rho$
0
ω_1



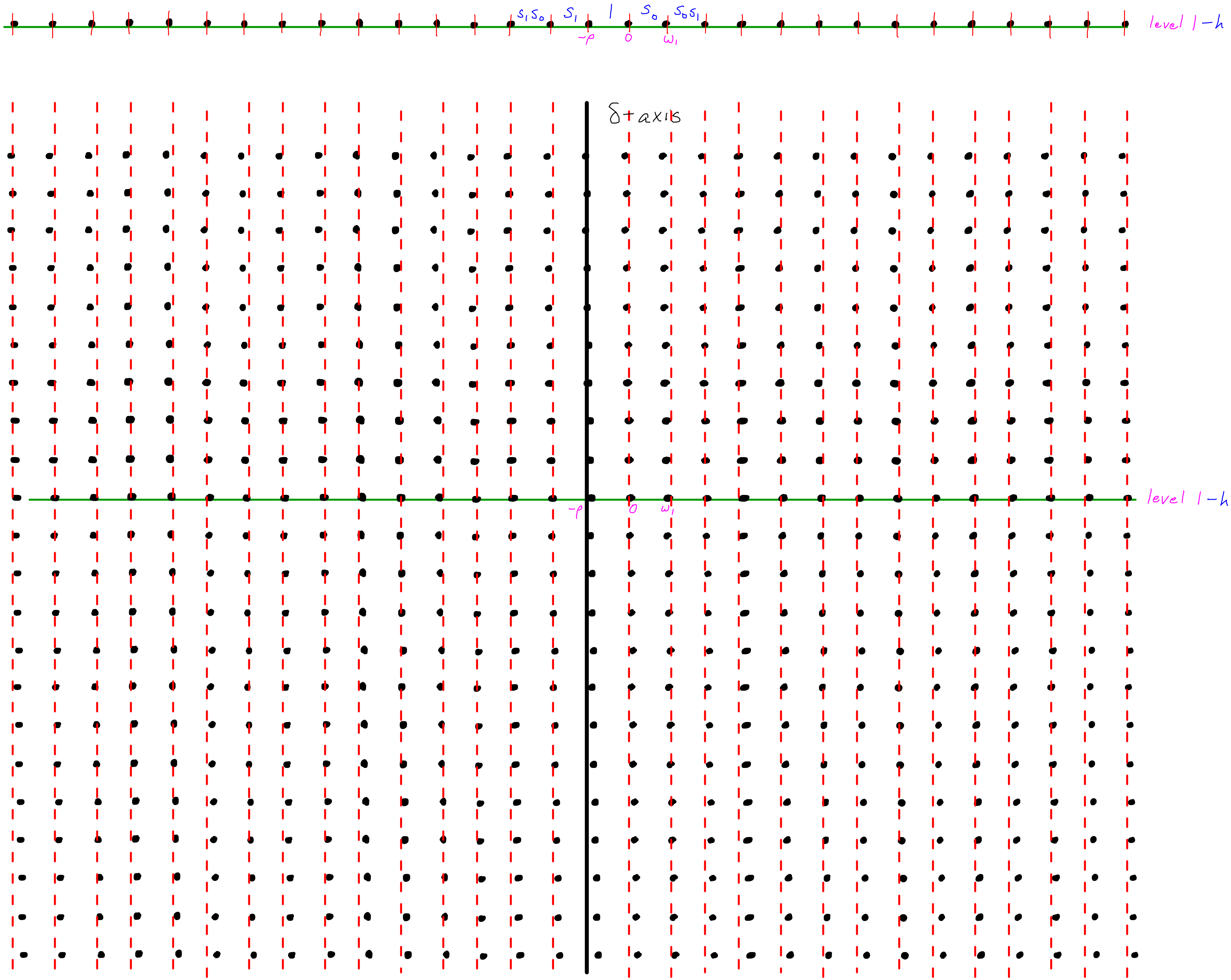
$s_1 s_0 s_1$
$s_1 s_0$
s_1
s_0
$s_0 s_1$
level $5-h$
$-\rho$
0
ω_1
δ-axis
level $5-h$
$-\rho$
0
ω_1



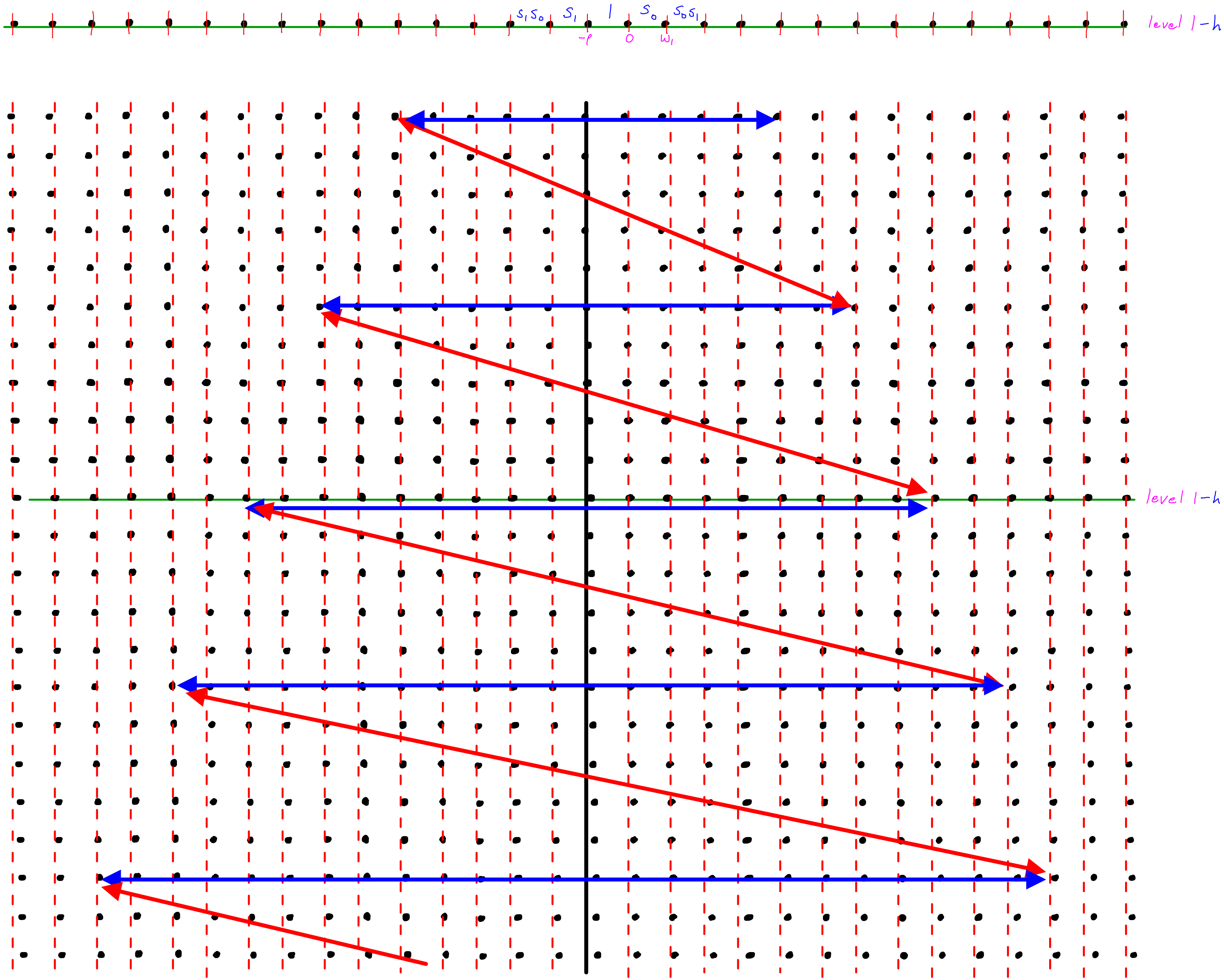
s1s0s1
s1s0
s1
1
s0
s0s1
level 5-h
-ρ
0
ω1
δ-axis
level 5-h
-ρ
0
ω1

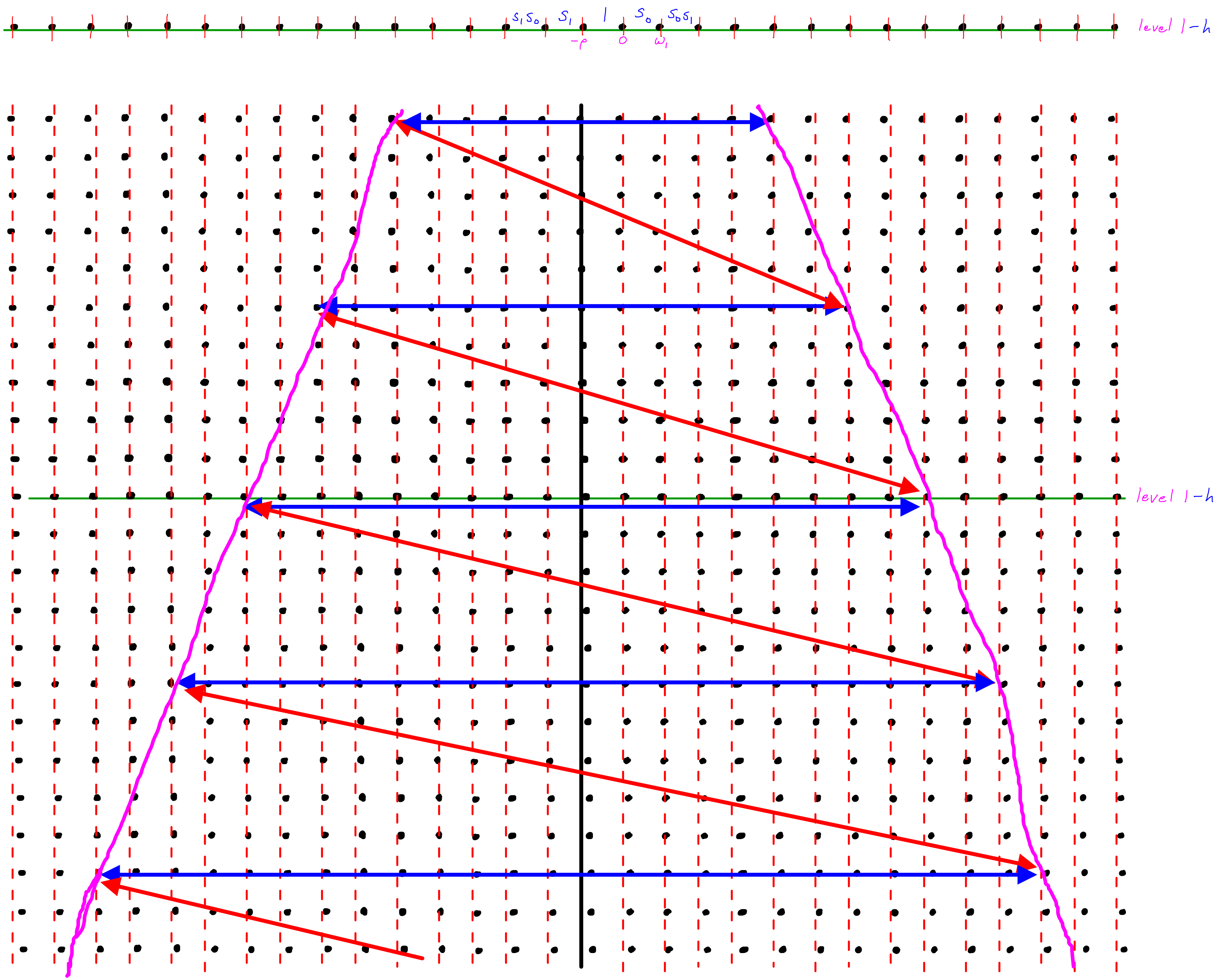


$s_1 s_0$
s_1
1
s_0
$-\rho$
0
ω_1
level 10-h
$s_1 s_0 s_1$
$s_1 s_0$
s_1
1
s_0
$s_0 s_1$
$-\rho$
0
ω_1
level 5-h
$s_1 s_0$
s_1
1
s_0
$s_0 s_1$
$-\rho$
0
ω_1
level 1-h
$-\rho$
0
ω_1
level 0-h
$s_0 s_1$
s_0
1
s_1
$s_1 s_0$
$s_1 s_0 s_1$
$-\rho$
0
ω_1
level -5-h
$s_0 s_1$
1
s_1
$s_1 s_0$
$-\rho$
0
ω_1
level -10-h

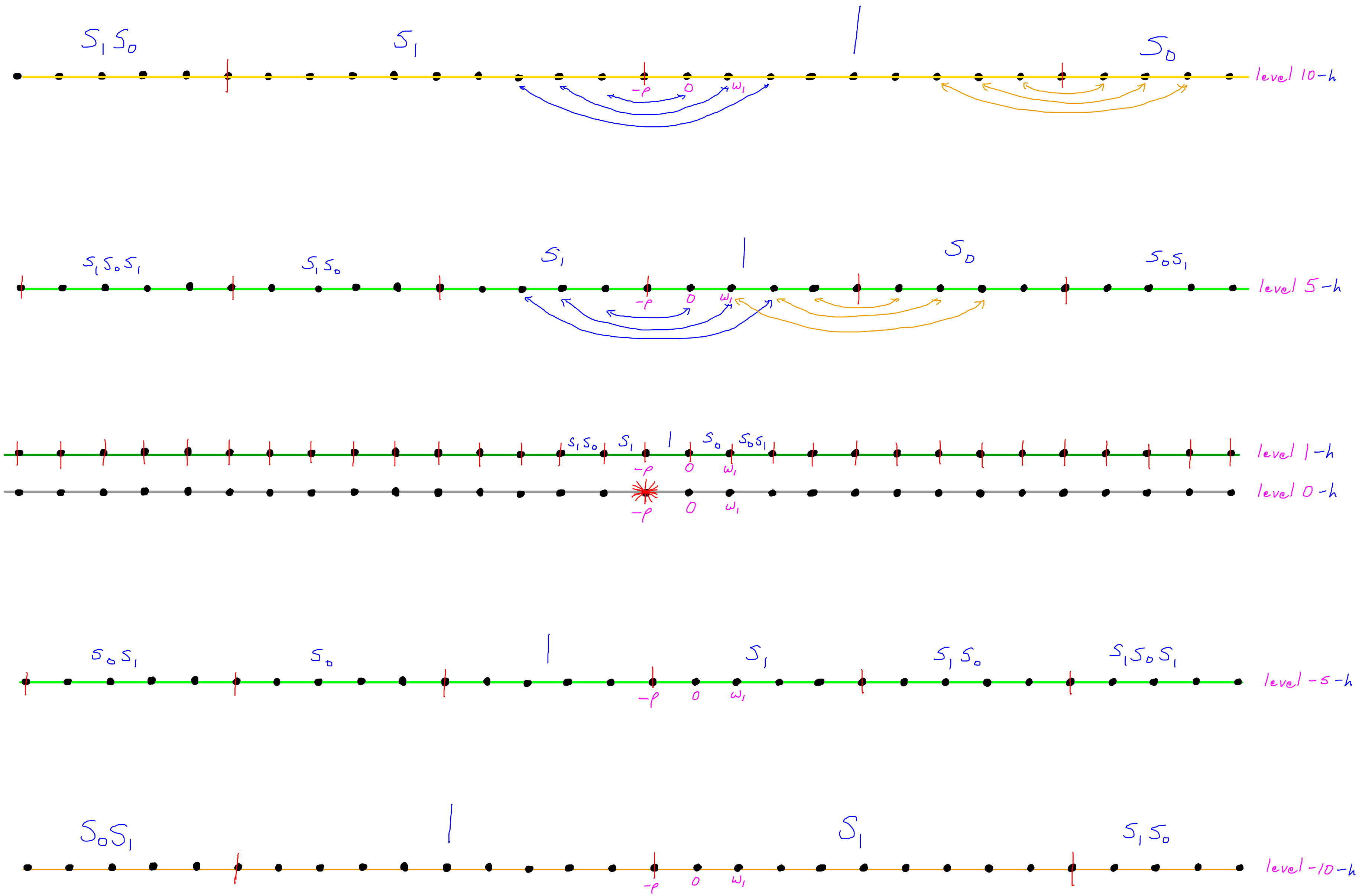


$s_1 s_0$
s_1
1
s_0
$s_0 s_1$
level $1-h$
$-\rho$
0
ω_1
δ-axis
level $1-h$
$-\rho$
0
ω_1

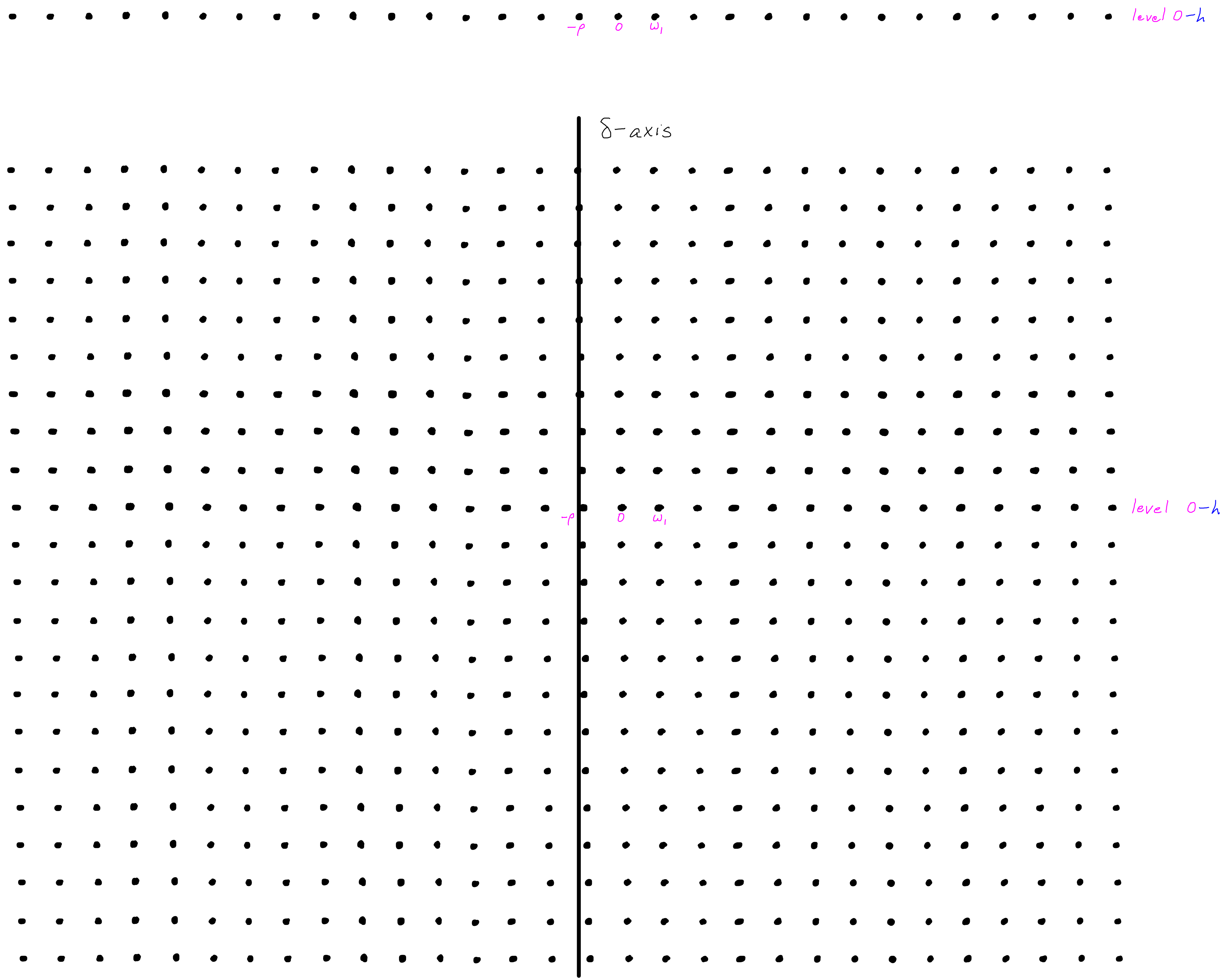

$s_1 s_0$
s_1
1
s_0
$s_0 s_1$
$-\rho$
0
ω_1
level $1-h$
level $1-h$

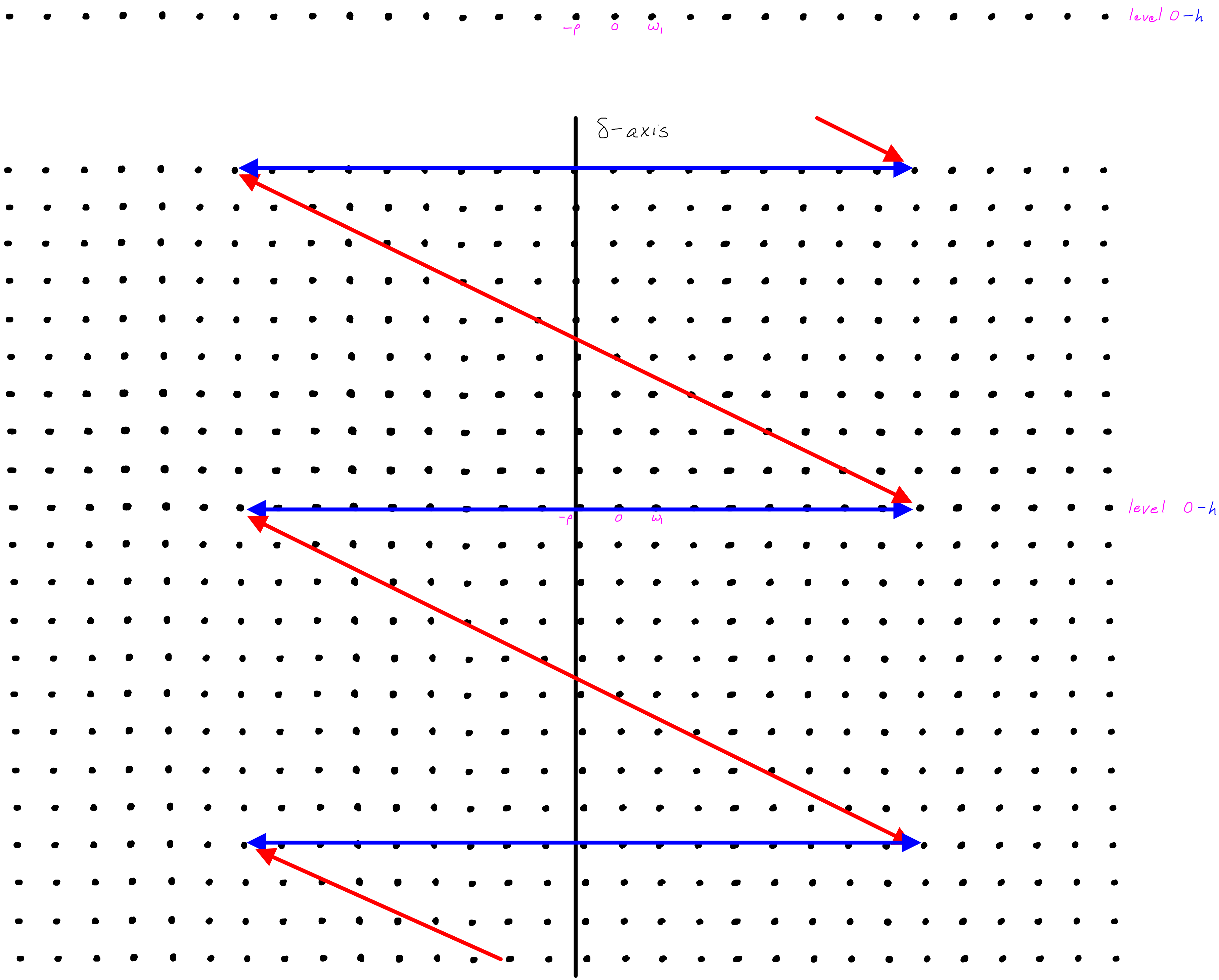


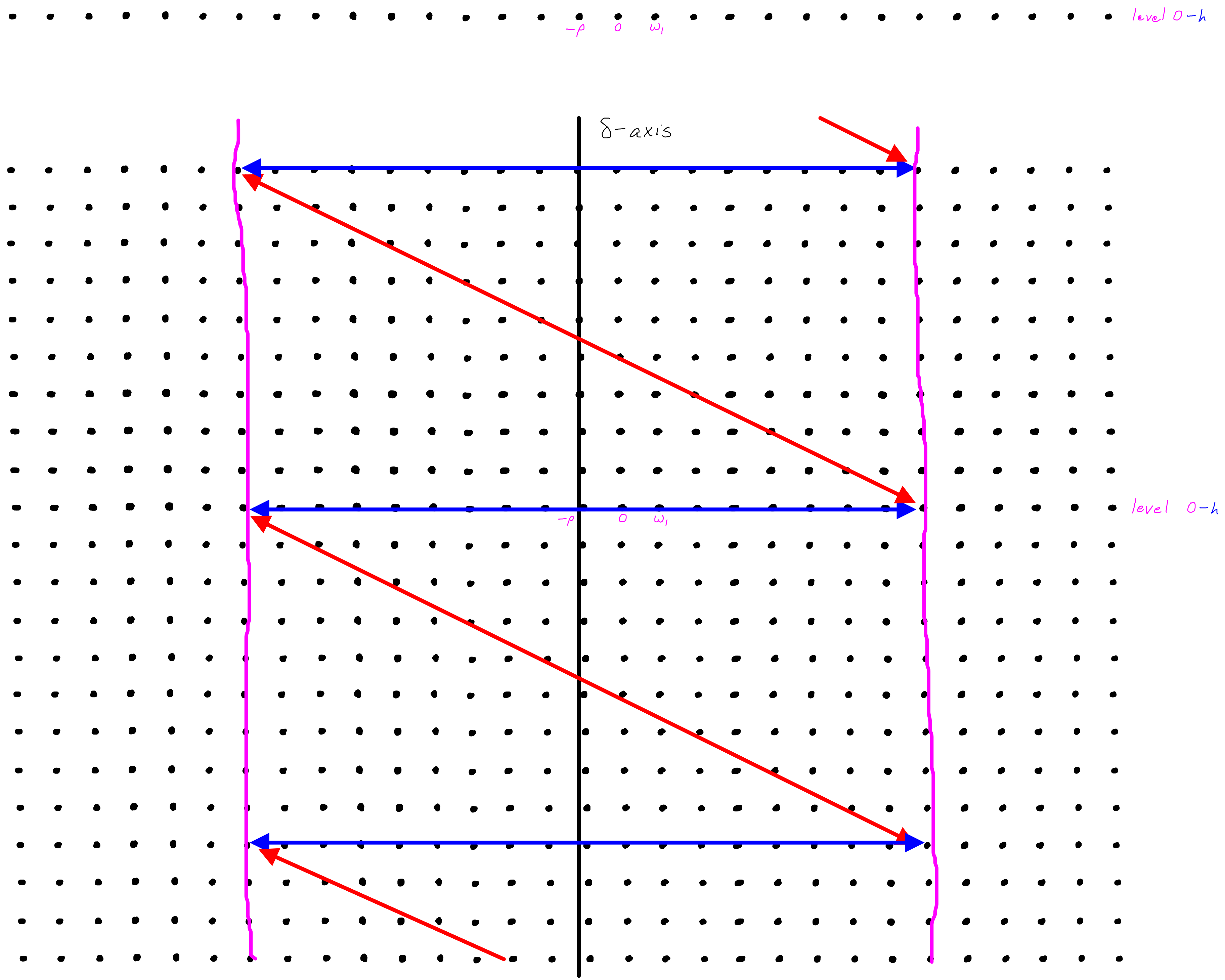
$s_1 s_0$
s_1
1
s_0
$s_0 s_1$
$-\rho$
0
ω_1
level $1-h$
level $1-h$



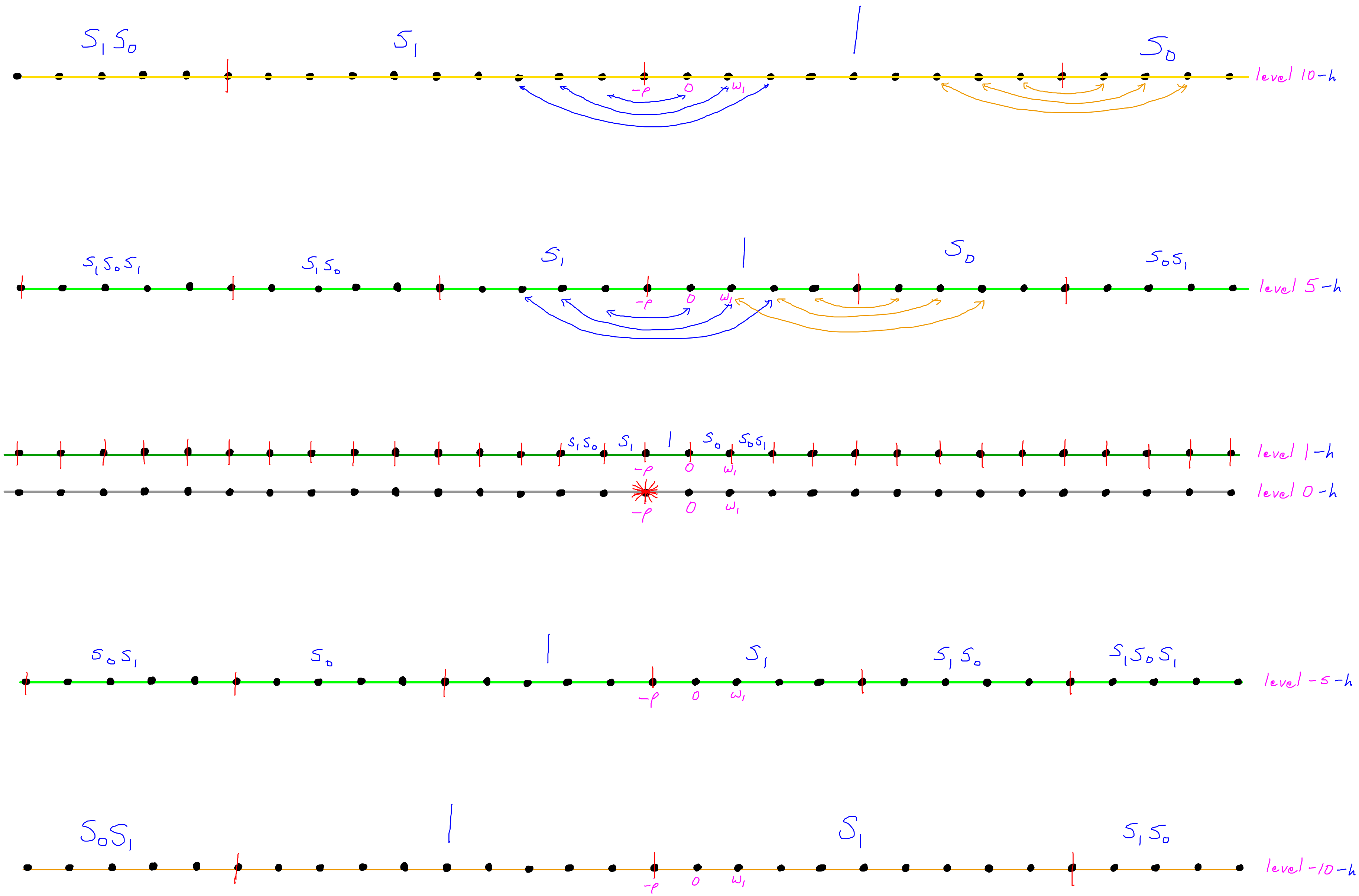
s1 s0
s1
1
s0
-ρ
0
ω1
level 10-h
s1 s0 s1
s1 s0
s1
1
s0
s0 s1
-ρ
0
ω1
level 5-h
s1 s0
s1
1
s0
s0 s1
-ρ
0
ω1
level 1-h
-ρ
0
ω1
level 0-h
s0 s1
s0
1
s1
s1 s0
s1 s0 s1
-ρ
0
ω1
level -5-h
s0 s1
1
s1
s1 s0
-ρ
0
ω1
level -10-h


level 0-h
-ρ
0
ω1
δ-axis
level 0-h
-ρ
0
ω1

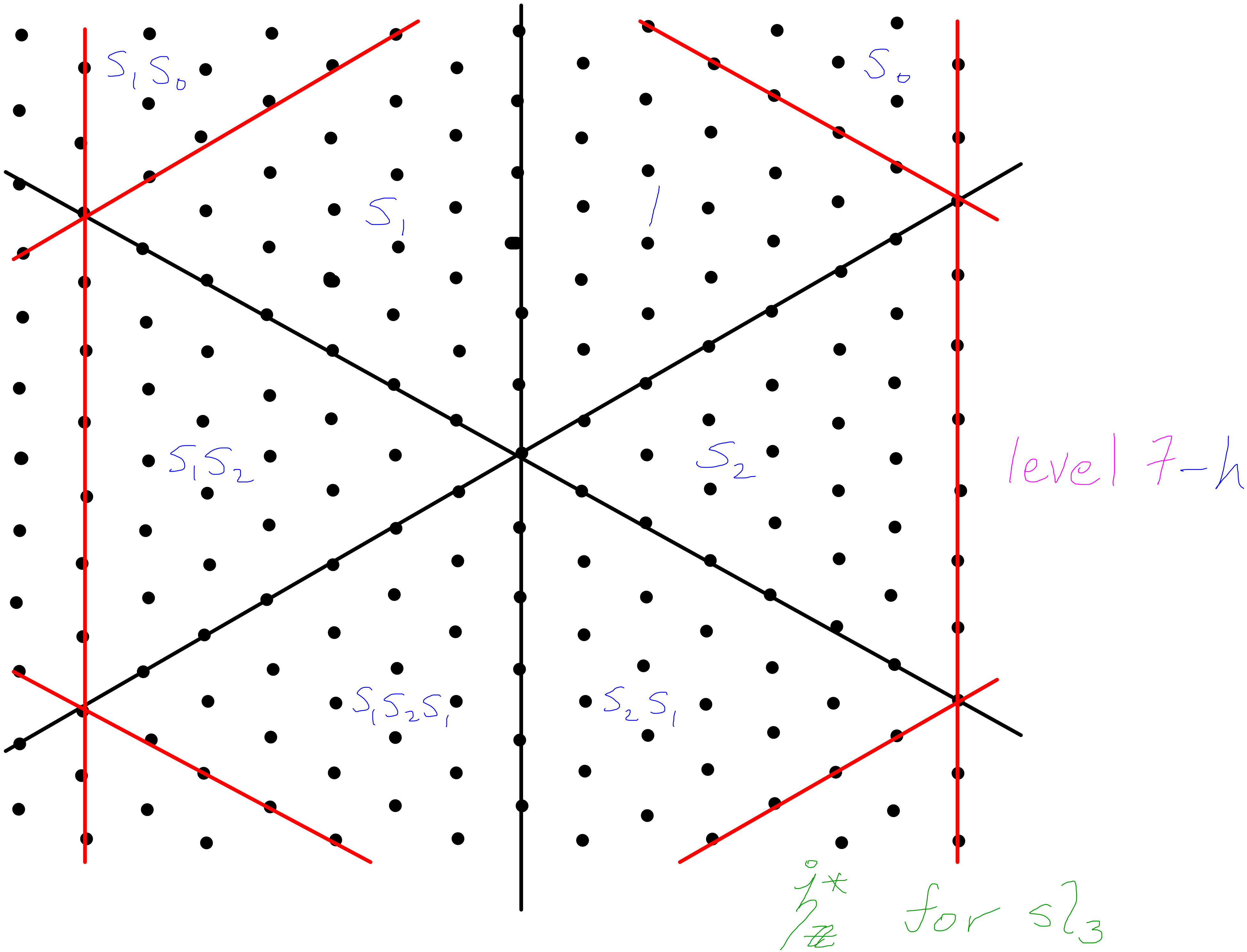

level 0 − h
−ρ
0
ω₁
δ-axis
level 0 − h
−ρ
0
ω₁



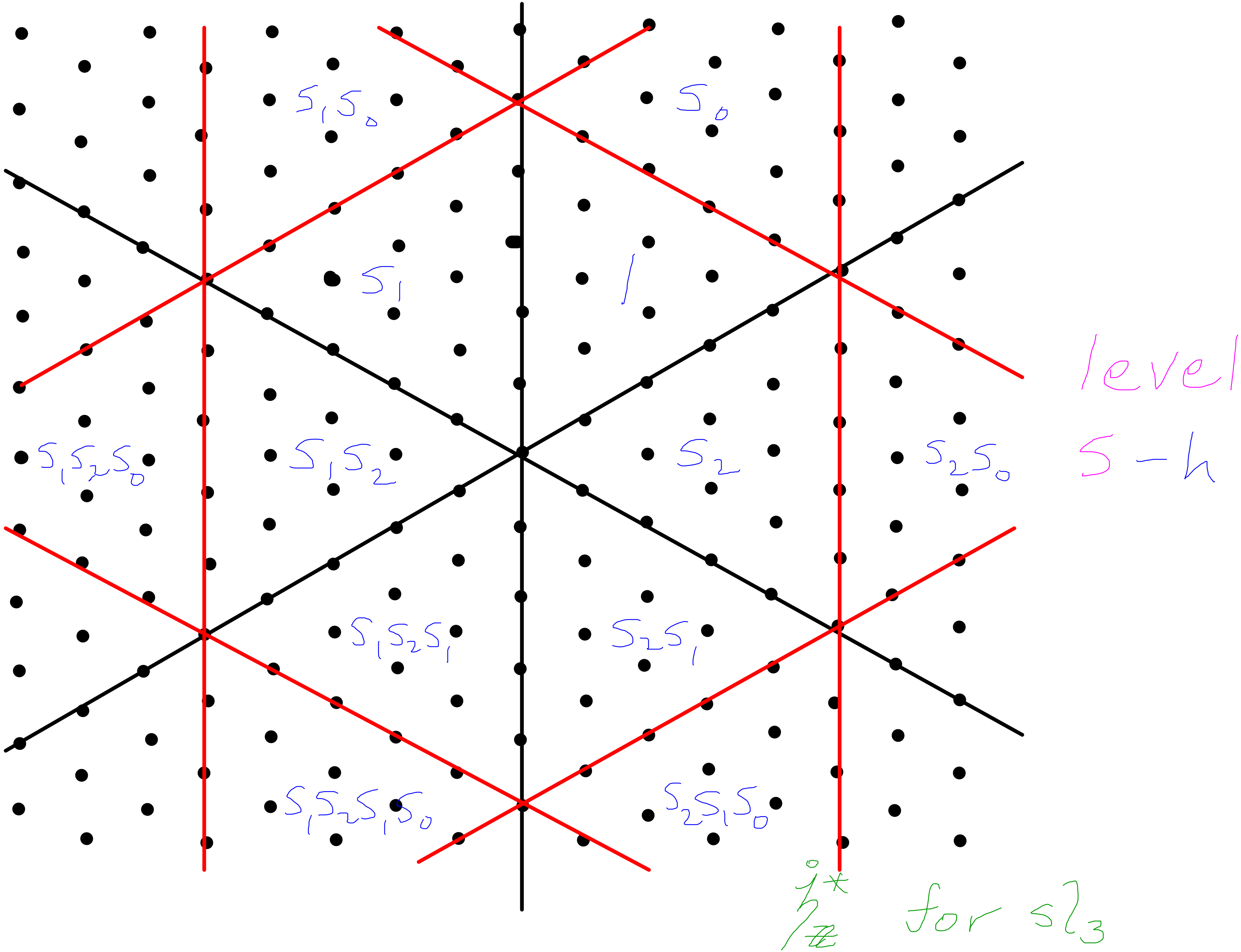
level 0−h
−ρ 0 ω₁
δ−axis
level 0−h
−ρ 0 ω₁



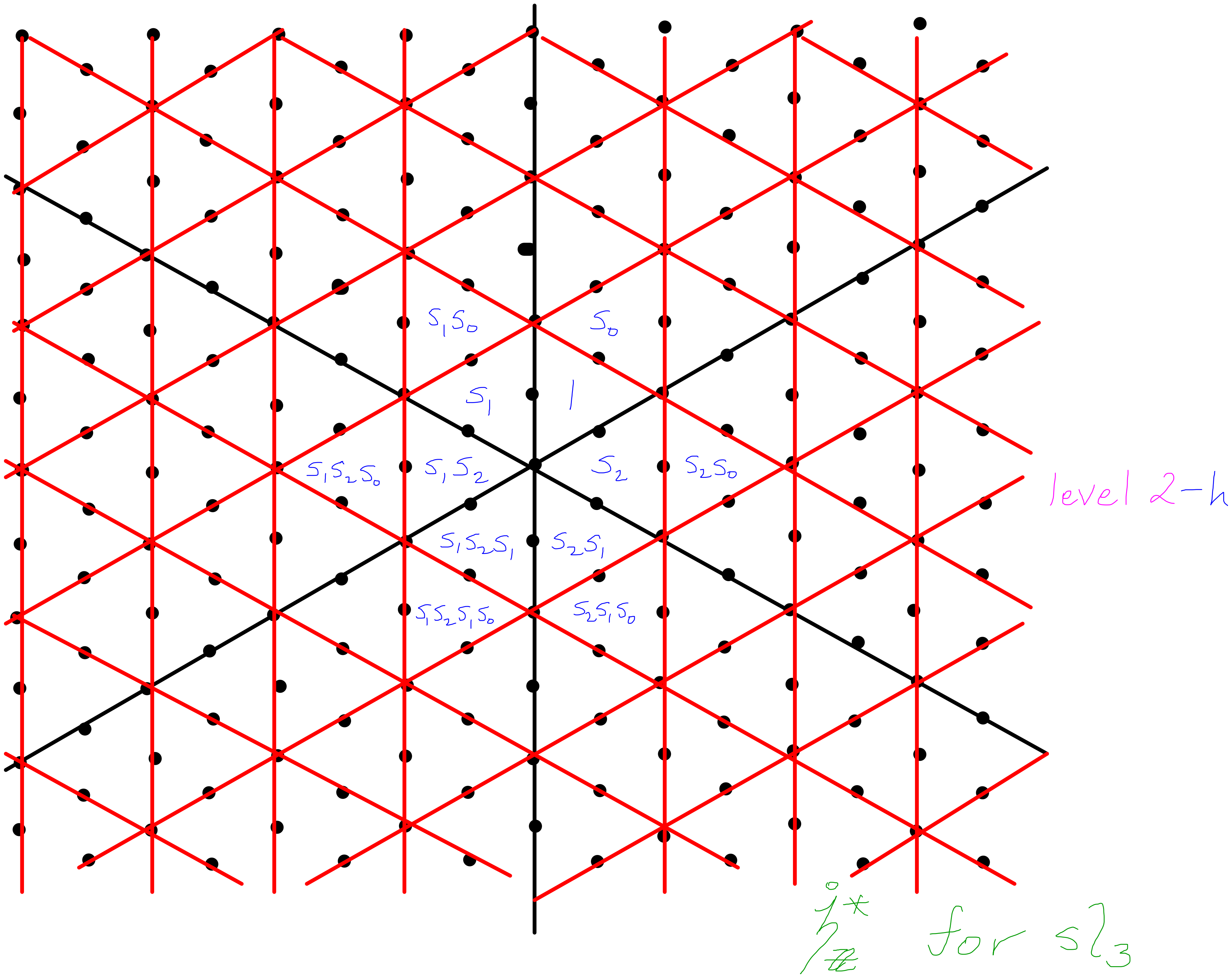
$s_1 s_0$ s_1 1 s_0
$-\rho$ 0 ω_1
level 10-h
$s_1 s_0 s_1$ $s_1 s_0$ s_1 1 s_0 $s_0 s_1$
$-\rho$ 0 ω_1
level 5-h
$s_1 s_0$ s_1 1 s_0 $s_0 s_1$
$-\rho$ 0 ω_1
level 1-h
$-\rho$ 0 ω_1
level 0-h
$s_0 s_1$ s_0 1 s_1 $s_1 s_0$ $s_1 s_0 s_1$
$-\rho$ 0 ω_1
level -5-h
$s_0 s_1$ 1 s_1 $s_1 s_0$
$-\rho$ 0 ω_1
level -10-h



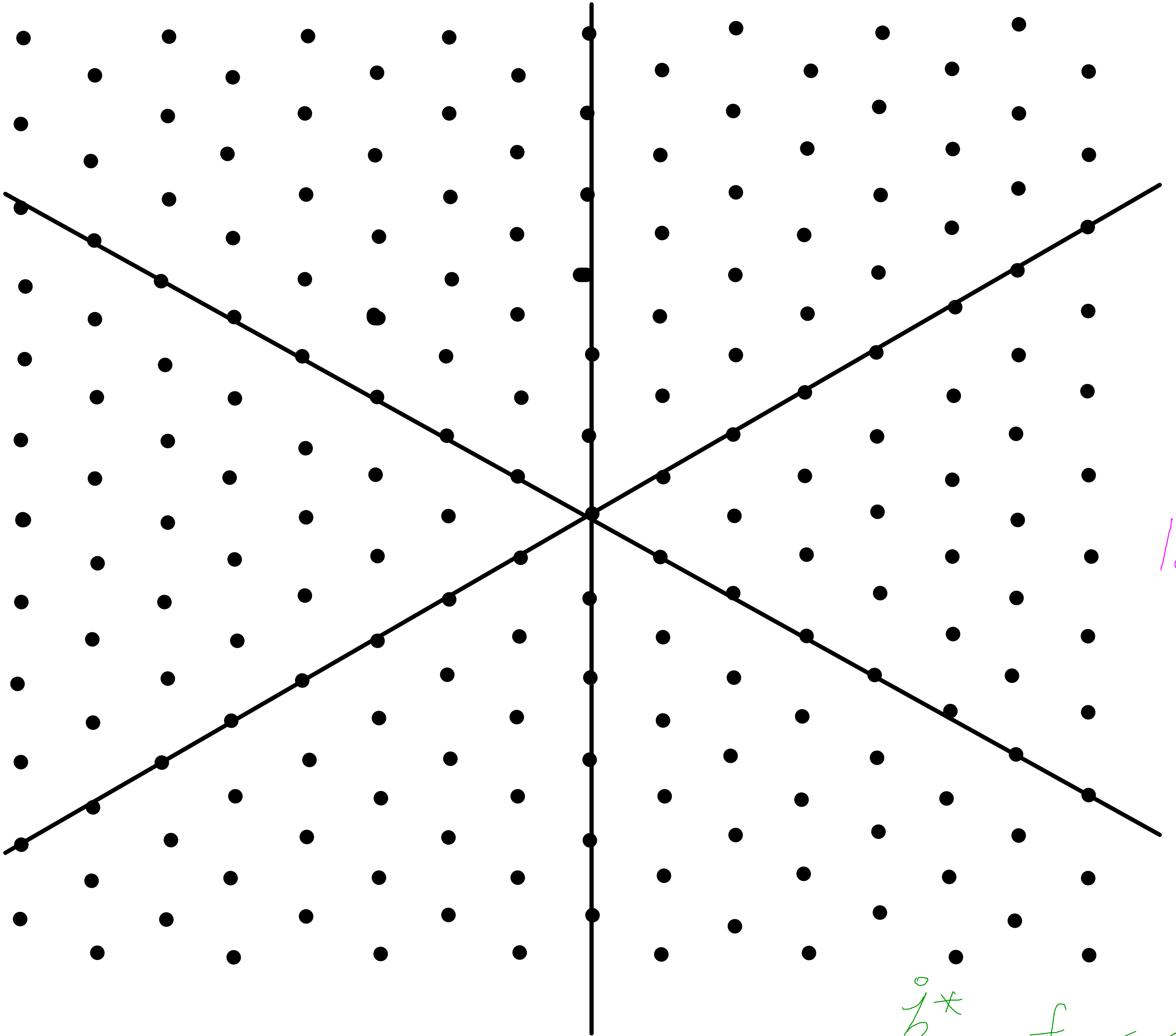
$s_1 s_0$
s_0
s_1
1
$s_1 s_2$
s_2
level 7-h
$s_1 s_2 s_1$
$s_2 s_1$
$\mathfrak{h}^*_{\mathbb{Z}}$ for $\mathfrak{sl}_3$



$s_1 s_0$
s_0
s_1
1
level 5-h
$s_1 s_2 s_0$
$s_1 s_2$
s_2
$s_2 s_0$
$s_1 s_2 s_1$
$s_2 s_1$
$s_1 s_2 s_1 s_0$
$s_2 s_1 s_0$
$\mathfrak{h}^*_{\mathbb{Z}}$ for $\mathfrak{sl}_3$

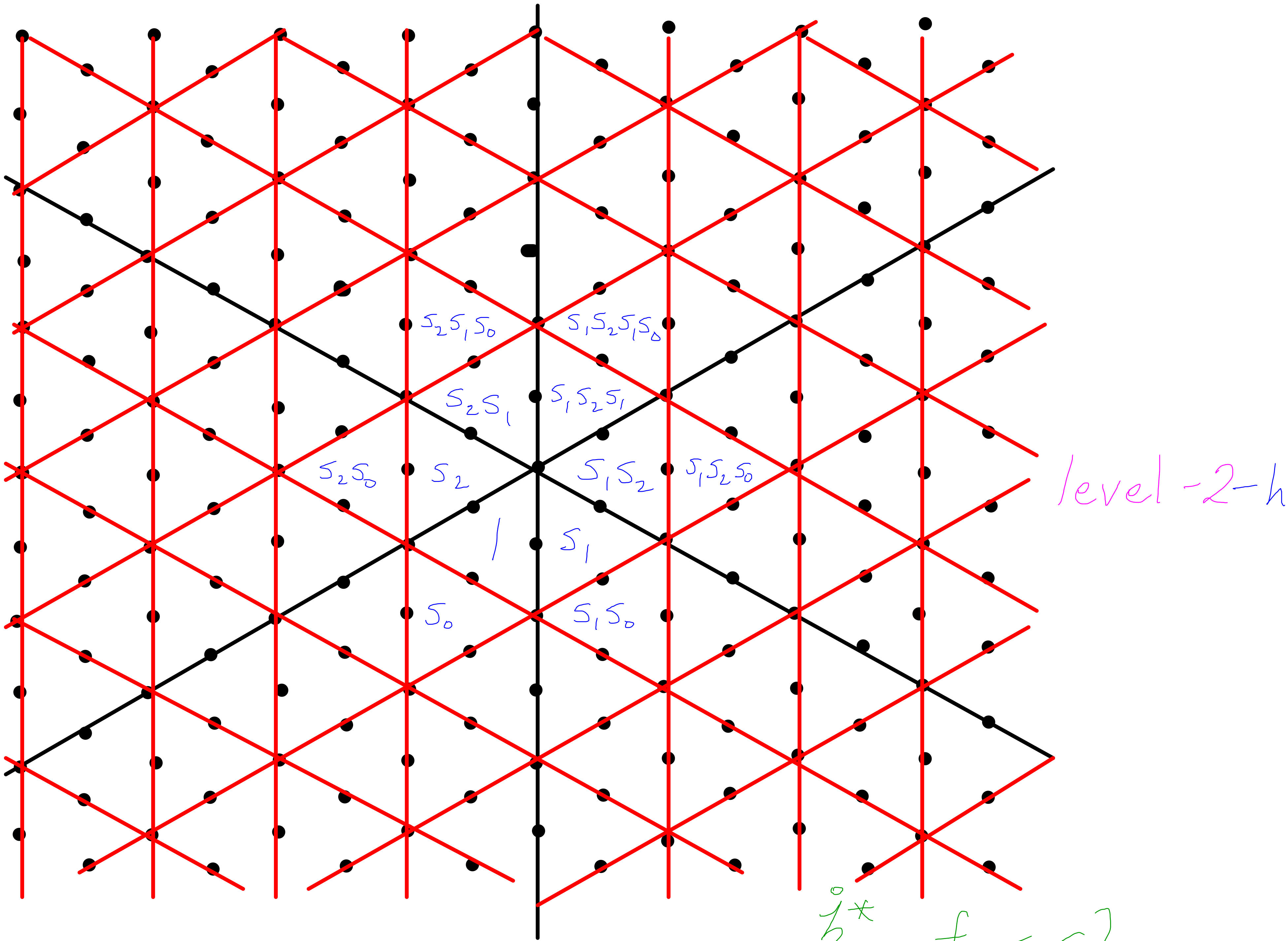


$s_1 s_0$
s_0
s_1
1
$s_1 s_2 s_0$
$s_1 s_2$
s_2
$s_2 s_0$
$s_1 s_2 s_1$
$s_2 s_1$
$s_1 s_2 s_1 s_0$
$s_2 s_1 s_0$
level 2-h
$\mathfrak{h}^*_{\mathbb{Z}}$ for $\mathfrak{sl}_3$

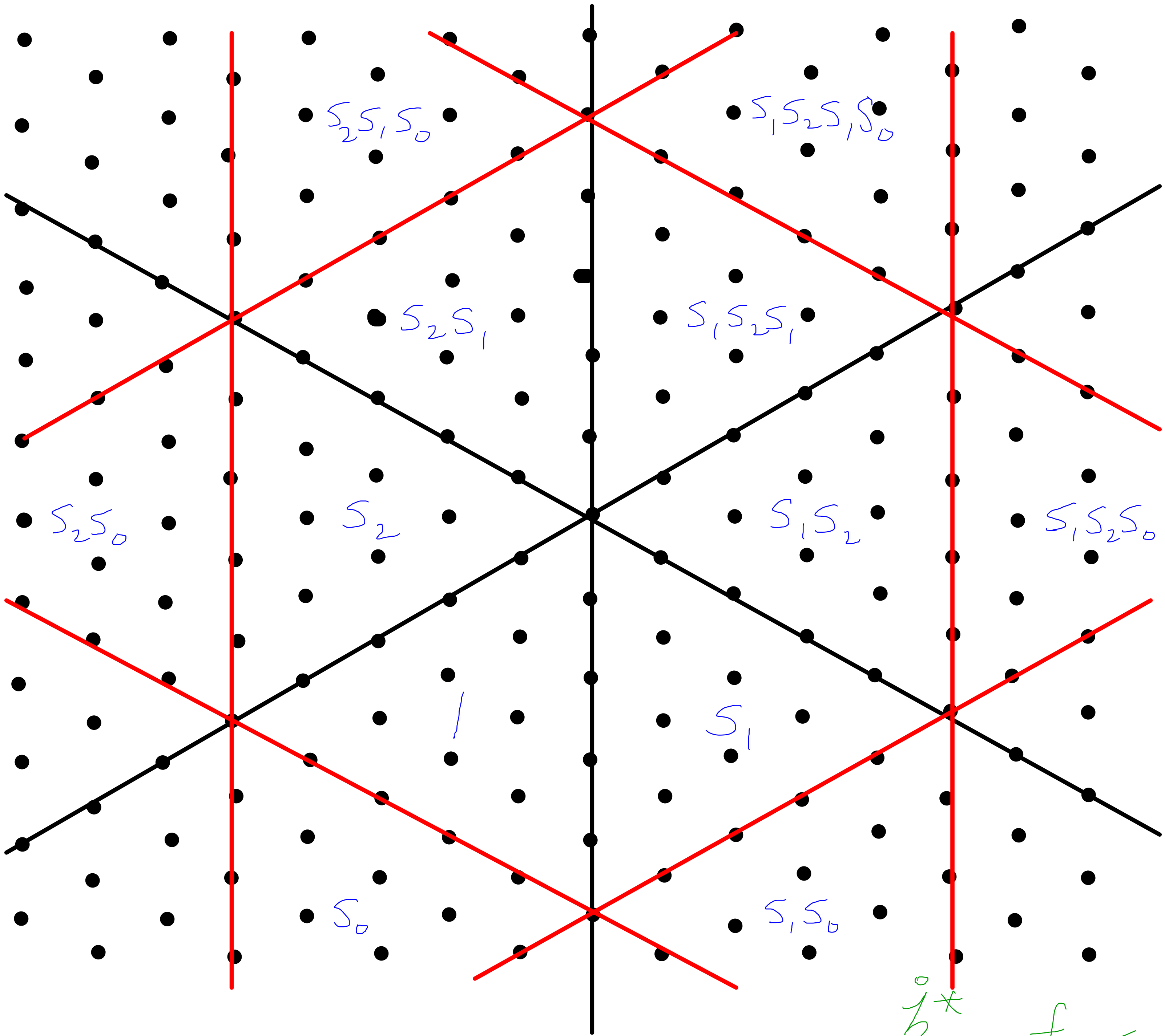


level 0-$\mathfrak{h}$

$\mathfrak{h}^*_{\mathbb{Z}}$ for $\mathfrak{sl}_3$

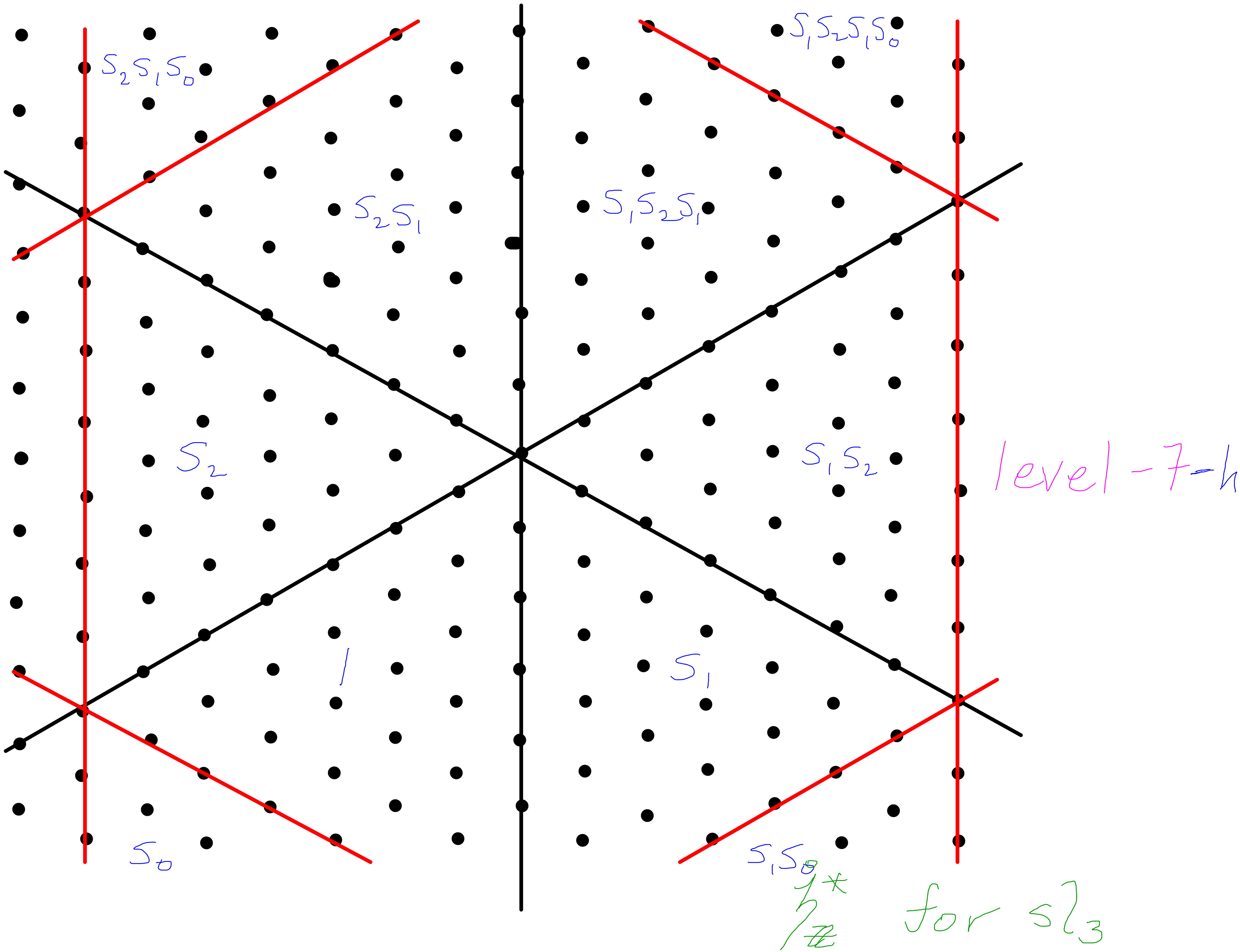


$\mathfrak{h}^*_{\mathbb{Z}}$ for $\mathfrak{sl}_3$

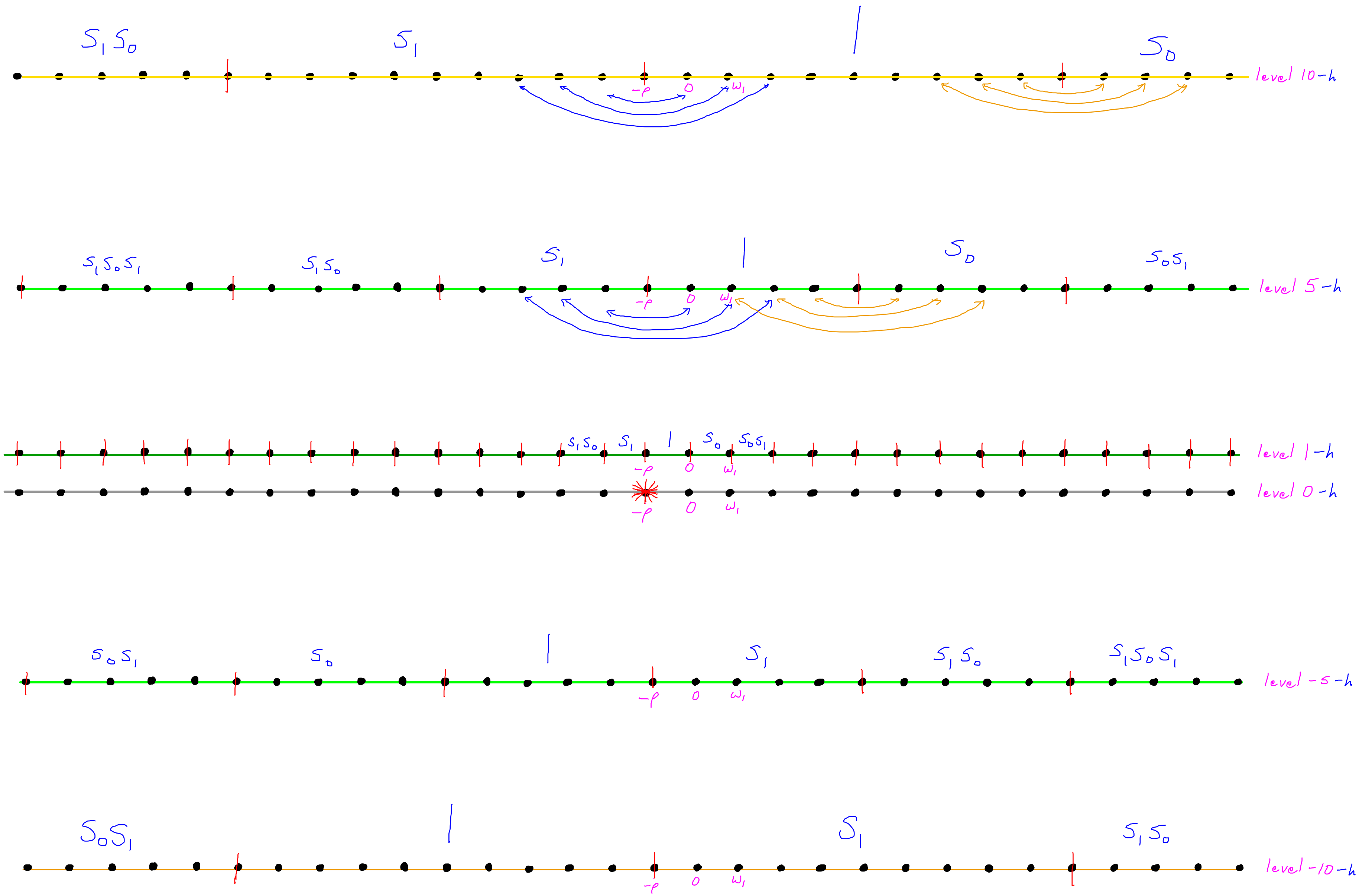


level $-5-h$

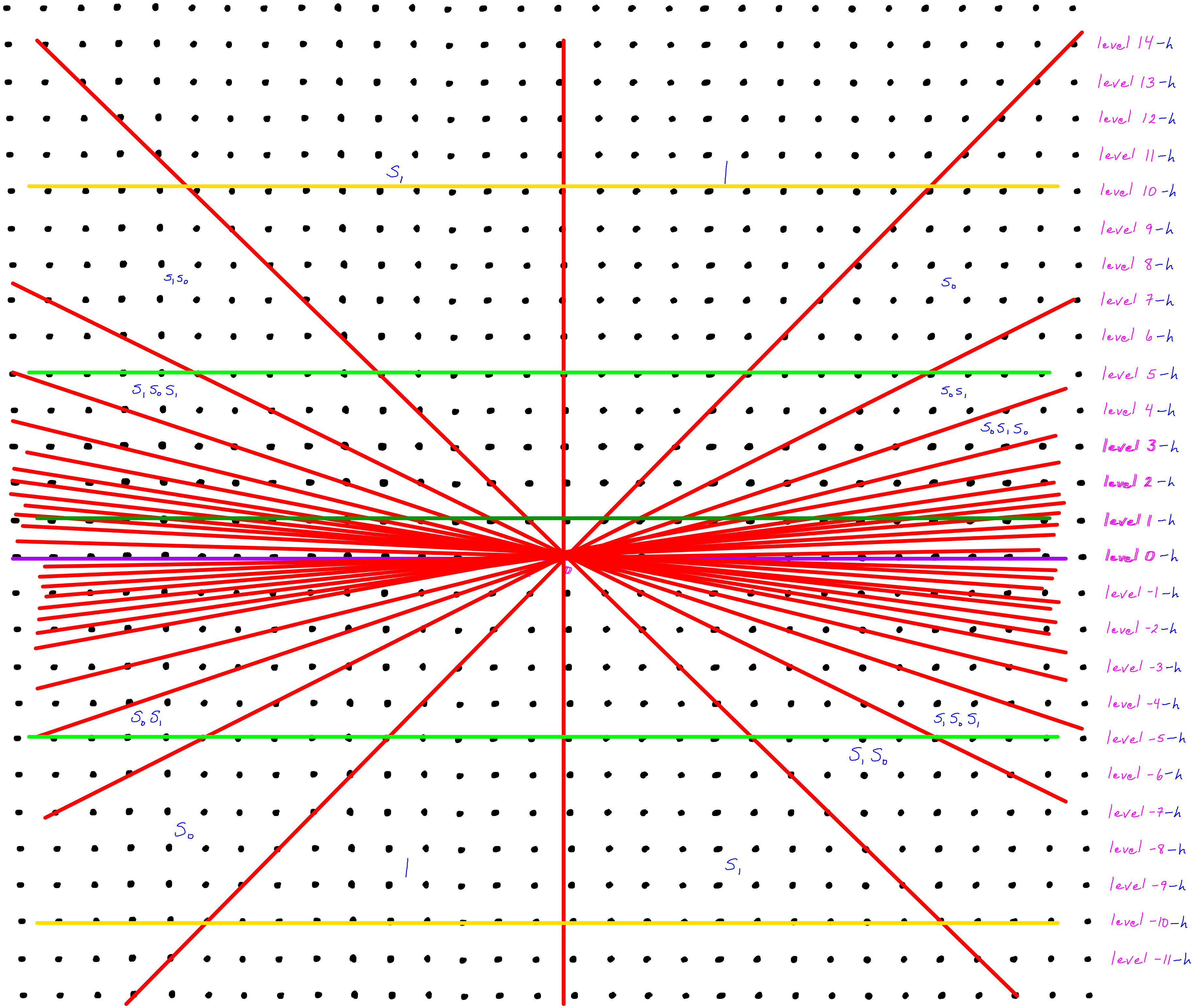
$\mathfrak{h}^*_{\mathbb{Z}}$ for $\mathfrak{sl}_3$



$s_2 s_1 s_0$
$s_1 s_2 s_1 s_0$
$s_2 s_1$
$s_1 s_2 s_1$
s_2
$s_1 s_2$
level-7-h
1
s_1
s_0
$s_1 s_0$
$\mathfrak{h}^*_{\mathbb{Z}}$ for $\mathfrak{sl}_3$



s1 s0
s1
1
s0
-ρ
0
ω1
level 10-h
s1 s0 s1
s1 s0
s1
1
s0
s0 s1
-ρ
0
ω1
level 5-h
s1 s0
s1
1
s0
s0 s1
-ρ
0
ω1
level 1-h
-ρ
0
ω1
level 0-h
s0 s1
s0
1
s1
s1 s0
s1 s0 s1
-ρ
0
ω1
level -5-h
s0 s1
1
s1
s1 s0
-ρ
0
ω1
level -10-h



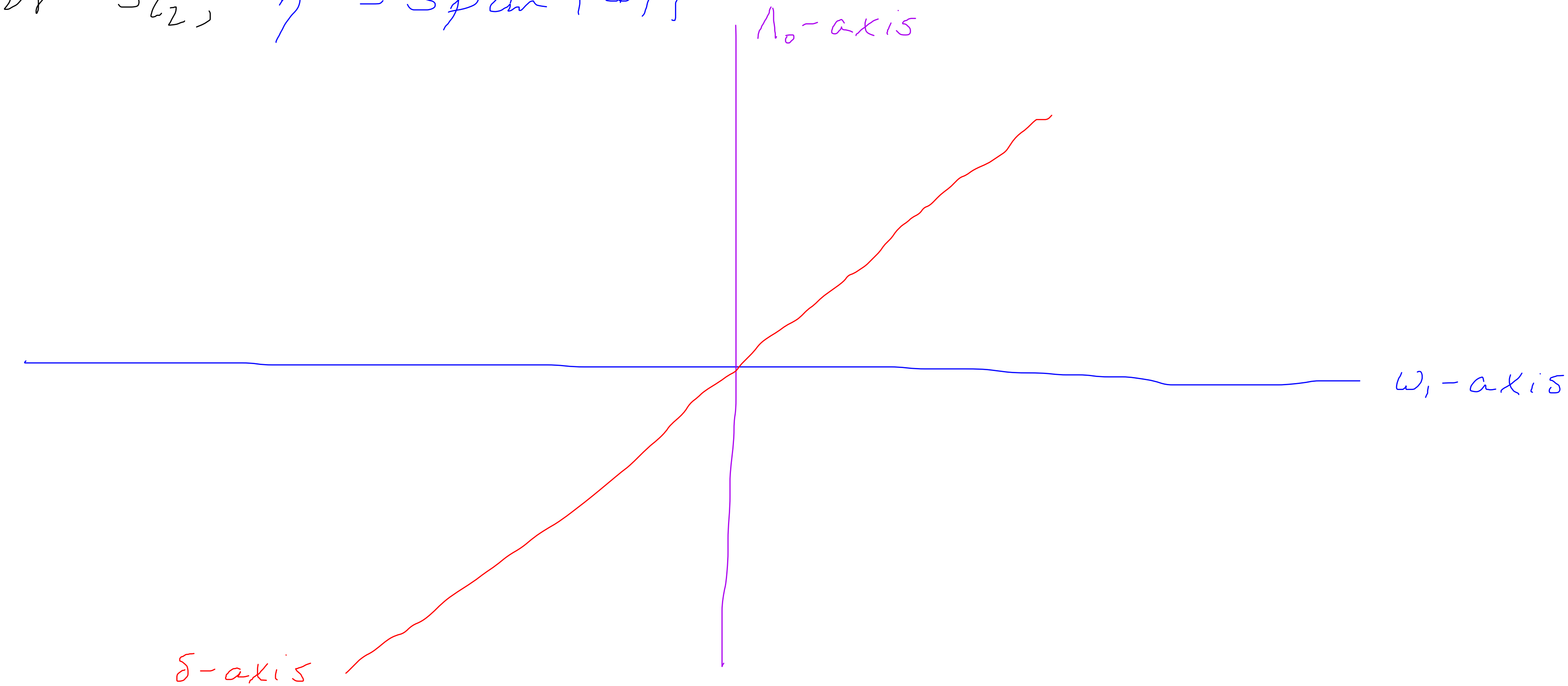
level 14-h
level 13-h
level 12-h
level 11-h
level 10-h
level 9-h
level 8-h
level 7-h
level 6-h
level 5-h
level 4-h
level 3-h
level 2-h
level 1-h
level 0-h
level -1-h
level -2-h
level -3-h
level -4-h
level -5-h
level -6-h
level -7-h
level -8-h
level -9-h
level -10-h
level -11-h
S_1
1
S_1S_0
S_0
$S_1S_0S_1$
S_0S_1
$S_0S_1S_0$
S_0S_1
$S_1S_0S_1$
S_1S_0
S_0
1
S_1

Formula (6.5.2) from Kac, Infinite dim. Lie algebras

$$t_\beta \lambda = \lambda + m\beta - \left(\bar{\lambda} + \tfrac{1}{2} m\beta \,|\, \beta\right)\delta \qquad \text{Here } \lambda \in \mathfrak{h}^*$$

$$\mathfrak{h}^* = \mathbb{C}\delta \oplus \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$$

For sl_2, $\mathring{\mathfrak{h}}^* = \operatorname{span}\{\omega_1\}$



joint work with

Martina Lanini

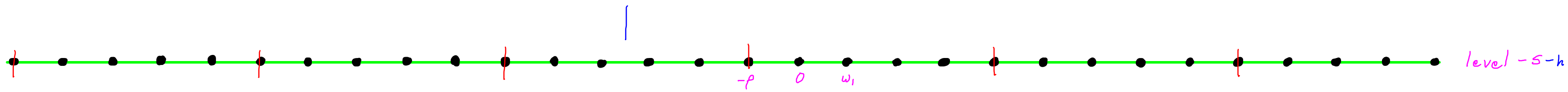
and

Paul Sobaje

Let $\ell \in \mathbb{Z}_{>0}$.

The level ℓ Fock space $\mathcal{F}_\ell$ is the $\mathbb{Z}[q,q^{-1}]$-module

generated by $\{ |\lambda\rangle \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}} \}$



with relations

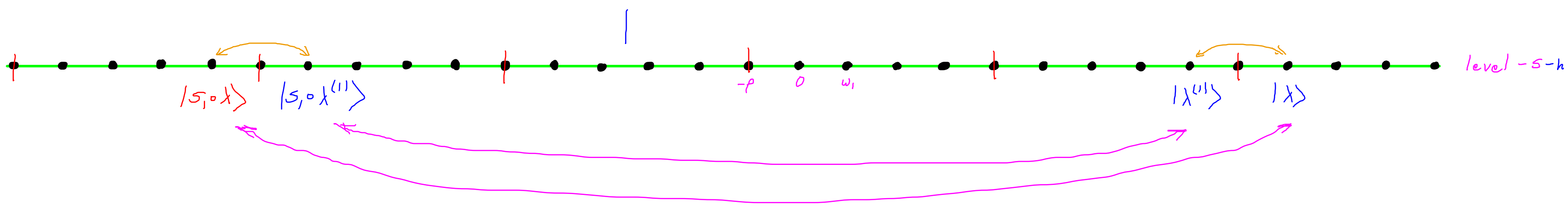
$$|s_i \circ \lambda\rangle = \begin{cases} -|\lambda\rangle, & \text{if } \langle \lambda+\rho, \alpha_i^\vee \rangle \in \ell\mathbb{Z}_{\geq 0} \\ -q|\lambda\rangle, & \text{if } 0 < \langle \lambda+\rho, \alpha_i^\vee \rangle < \ell \\ -q|s_i \circ \lambda^{(1)}\rangle - |\lambda^{(1)}\rangle - q|\lambda\rangle, & \text{otherwise} \end{cases}$$

for $i \in \{1, 2, \ldots, n\}$

Let $\ell \in \mathbb{Z}_{>0}$.

The level ℓ Fock space $\mathcal{F}_\ell$ is the $\mathbb{Z}[q,q^{-1}]$-module

generated by $\{ |\lambda\rangle \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}} \}$



with relations

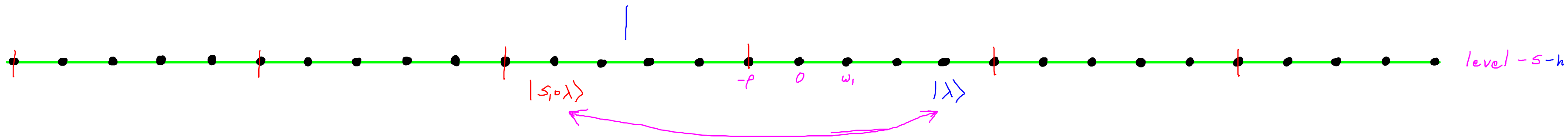
$$|s_i \circ \lambda\rangle = \begin{cases} -|\lambda\rangle, & \text{if } \langle \lambda+\rho, \alpha_i^\vee \rangle \in \ell\mathbb{Z}_{\geq 0} \\ -q|\lambda\rangle, & \text{if } 0 < \langle \lambda+\rho, \alpha_i^\vee \rangle < \ell \\ -q|s_i \circ \lambda^{(1)}\rangle - |\lambda^{(1)}\rangle - q|\lambda\rangle, & \text{otherwise} \end{cases}$$

for $i \in \{1, 2, \dots, n\}$

Let $\ell \in \mathbb{Z}_{>0}$.

The level ℓ Fock space $\mathcal{F}_\ell$ is the $\mathbb{Z}[q,q^{-1}]$-module

generated by $\{ |\lambda\rangle \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}} \}$



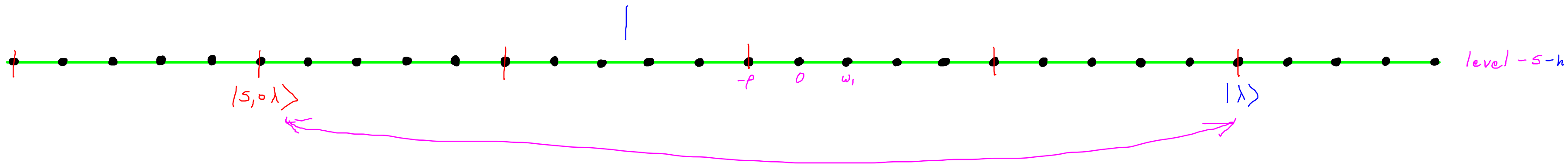
with relations

$$|s_i \circ \lambda\rangle = \begin{cases} -|\lambda\rangle, & \text{if } \langle \lambda+\rho, \alpha_i^\vee \rangle \in \ell\mathbb{Z}_{\geq 0} \\ -q|\lambda\rangle, & \text{if } 0 < \langle \lambda+\rho, \alpha_i^\vee \rangle < \ell \\ -q|s_i\circ\lambda^{(1)}\rangle - |\lambda^{(1)}\rangle - q|\lambda\rangle, & \text{otherwise} \end{cases}$$

for $i \in \{1, 2, \ldots, n\}$

Let $\ell \in \mathbb{Z}_{>0}$.

The level ℓ Fock space $\mathcal{F}_\ell$ is the $\mathbb{Z}[q,q^{-1}]$-module generated by $\{ |\lambda\rangle \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}} \}$



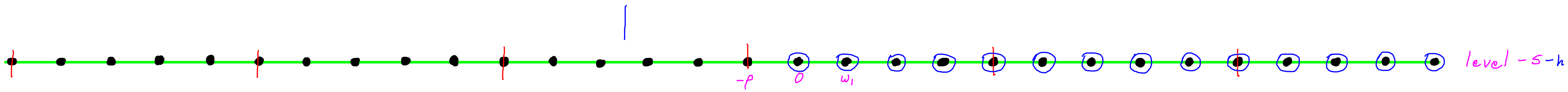
with relations

$$|s_i \circ \lambda\rangle = \begin{cases} -|\lambda\rangle, & \text{if } \langle \lambda+\rho, \alpha_i^\vee \rangle \in \ell\mathbb{Z}_{\geq 0} \\ -q|\lambda\rangle, & \text{if } 0 < \langle \lambda+\rho, \alpha_i^\vee \rangle < \ell \\ -q|s_i \circ \lambda^{(1)}\rangle - |\lambda^{(1)}\rangle - q|\lambda\rangle, & \text{otherwise} \end{cases}$$

for $i \in \{1, 2, \ldots, n\}$

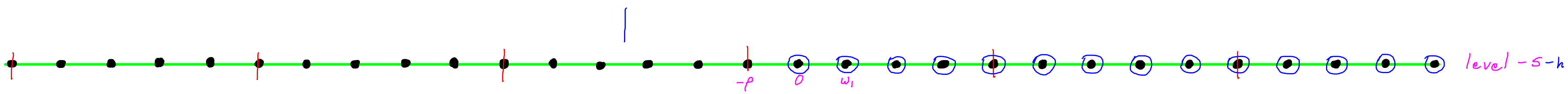
Let $\ell \in \mathbb{Z}_{>0}$. The level ℓ Fock space $\mathcal{F}_\ell$

has $\mathbb{Z}[q,q^{-1}]$-basis $\{ |\lambda\rangle \mid \lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+ \}$



Let $\ell \in \mathbb{Z}_{>0}$. The level ℓ Fock space $\mathcal{F}_\ell$

has $\mathbb{Z}[q,q^{-1}]$-basis $\left\{ |\lambda\rangle \mid \lambda \in \left(\mathfrak{h}^{\circ *}_{\mathbb{Z}}\right)^{+} \right\}$



The bar involution $\bar{\ }: \mathcal{F}_\ell \to \mathcal{F}_\ell$ is the $\mathbb{Z}$-linear map

$$\bar{q} = q^{-1} \quad \text{and} \quad \overline{|\lambda\rangle} = q^{\ell(w_\lambda)} (-q^{-1})^{\ell(w_0)} |w_0 \circ \lambda\rangle$$

where

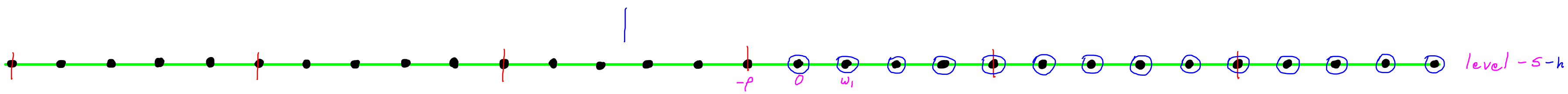
w_0 is the longest element of W_0, and

w_λ is the longest element of $W_\lambda = \mathrm{Stab}_{W_0 \circ}(\lambda)$,

the stabilizer of λ under the dot action of W_0.

Let $\ell \in \mathbb{Z}_{>0}$. The level ℓ Fock space $\mathcal{F}_\ell$

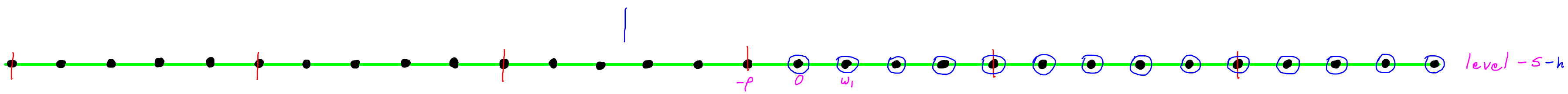
has $\mathbb{Z}[q, q^{-1}]$-basis $\{ |\lambda\rangle \mid \lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+ \}$



$\overline{q} = q^{-1}$ and $\overline{|\lambda\rangle} = q^{\ell(w_\lambda)} (-q^{-1})^{\ell(w_0)} |w_0 \circ \lambda\rangle$

Let $\ell \in \mathbb{Z}_{>0}$. The level ℓ Fock space $\mathcal{F}_\ell$

has $\mathbb{Z}[q,q^{-1}]$-basis $\{\,|\lambda\rangle \mid \lambda \in (\mathfrak{h}^{\circ *}_{\mathbb{Z}})^+\}$



$$\bar{q} = q^{-1} \quad \text{and} \quad \overline{|\lambda\rangle} = q^{\ell(w_\lambda)}(-q^{-1})^{\ell(w_0)}\,|w_0 \circ \lambda\rangle$$

Define $C_\lambda \in \mathcal{F}_\ell$ by

$$\overline{C_\lambda} = C_\lambda \quad \text{and} \quad C_\lambda = |\lambda\rangle + \sum_{\mu \neq \lambda} p_{\mu\lambda}\,|\mu\rangle$$

with $p_{\lambda\mu} \in q\mathbb{Z}[q]$.

Theorem

$$\text{Grothendieck group}\begin{pmatrix}\text{finite dimensional}\\ U_\epsilon \mathfrak{g}\text{-modules}\\ \epsilon^\ell = 1\end{pmatrix} \xrightarrow{\sim} \mathcal{F}_\ell$$

$$[\Delta_\epsilon(\lambda)] \longmapsto |\lambda\rangle$$

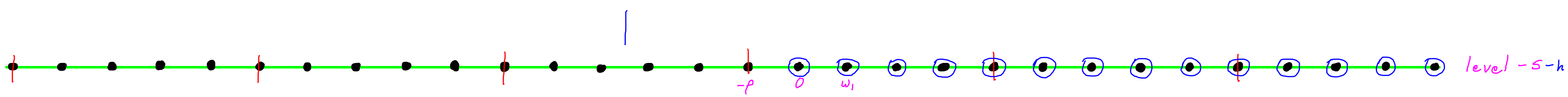
$$[L_\epsilon(\lambda)] \longmapsto C_\lambda$$

Where

$\Delta_\epsilon(\lambda)$ is the Weyl module (the irreducible for generic ϵ)

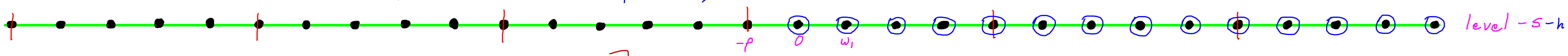
$L_\epsilon(\lambda)$ is the simple module

of highest weight λ.



Theorem

$$\text{Grothendieck group}\left(\begin{matrix}\text{finite dimensional}\\ U_\varepsilon\mathring{\mathfrak{g}}\text{-modules}\\ \varepsilon^\ell=1\end{matrix}\right) \xrightarrow{\sim} \mathcal{F}_\ell$$



$$[\Delta_\varepsilon(\lambda)] \longmapsto |\lambda\rangle$$

$$[L_\varepsilon(\lambda)] \longmapsto C_\lambda$$

More precisely,

$\Delta_\varepsilon(\lambda)$ has a Jantzen filtration

$$\Delta_\varepsilon(\lambda) = \Delta_\varepsilon(\lambda)^{(0)} \supseteq \Delta_\varepsilon(\lambda)^{(1)} \supseteq \Delta_\varepsilon(\lambda)^{(2)} \supseteq \cdots$$

and

$$|\lambda\rangle = \sum_{\mu\in(\mathring{\mathfrak{h}}_{\mathbb{Z}}^*)^+} \left(\sum_{j\in\mathbb{Z}_{\geq 0}} q^j \dim\left(\mathrm{Hom}\left(\frac{\Delta_\varepsilon(\lambda)^{(j)}}{\Delta_\varepsilon(\lambda)^{(j+1)}}, L_\varepsilon(\mu)\right)\right)\right) C_\mu$$

Let $\ell \in \mathbb{Z}_{>0}$. The level ℓ Fock space $\mathcal{F}_\ell$

has $\mathbb{Z}[q,q^{-1}]$-basis $\{ |\lambda\rangle \mid \lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+ \}$

$$\bar{q} = q^{-1} \quad \text{and} \quad \overline{|\lambda\rangle} = q^{\ell(w_\lambda)}(-q^{-1})^{\ell(w_0)} |w_0 \circ \lambda\rangle$$

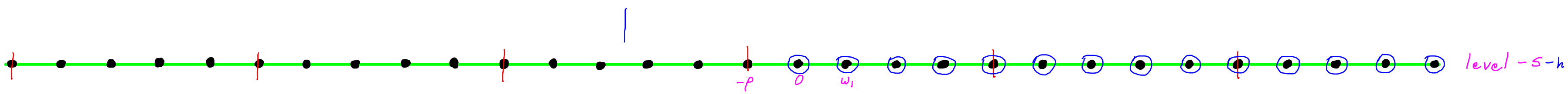
$$\overline{C_\lambda} = C_\lambda \quad \text{and} \quad C_\lambda = |\lambda\rangle + \sum_{\mu \neq \lambda} p_{\mu\lambda} |\mu\rangle$$

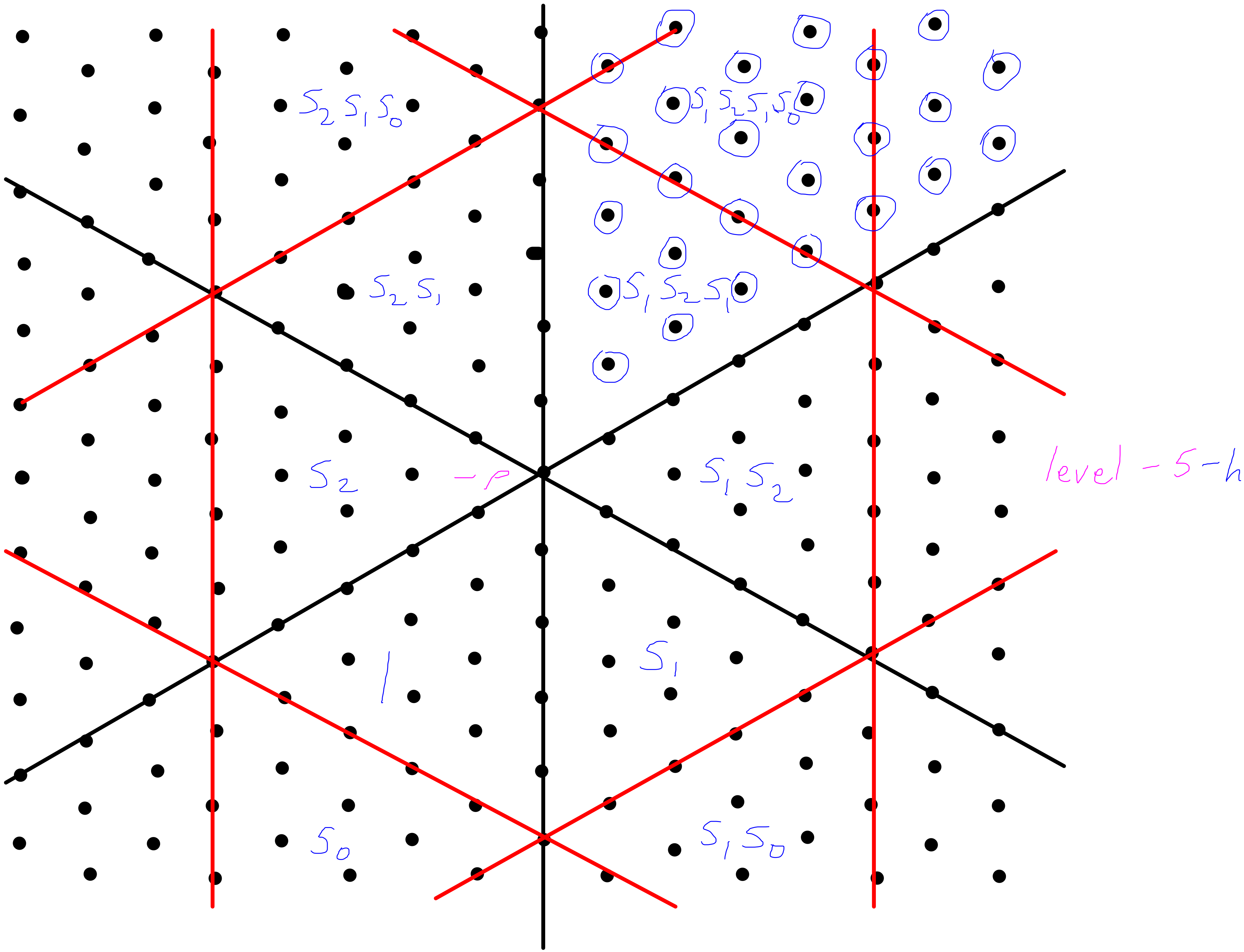
with $p_{\lambda\mu} \in q\mathbb{Z}[q]$.

$$\text{Grothendieck group}\begin{pmatrix} \text{finite dimensional} \\ U_\epsilon \mathring{\mathfrak{g}}\text{-modules} \\ \epsilon^\ell = 1 \end{pmatrix} \xrightarrow{\sim} \mathcal{F}_\ell$$

$$[\Delta_\epsilon(\lambda)] \longmapsto |\lambda\rangle$$

$$[L_\epsilon(\lambda)] \longmapsto C_\lambda$$





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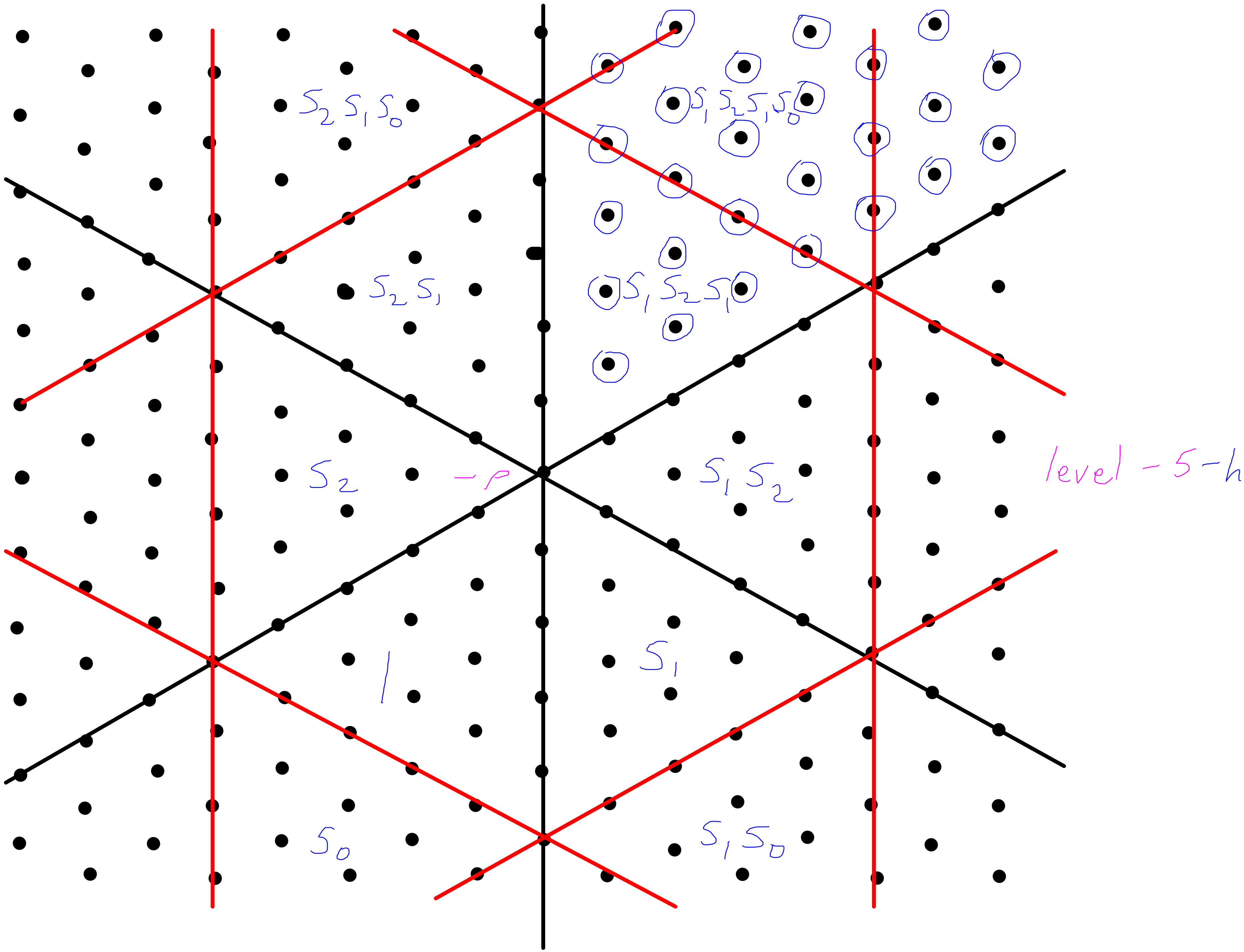
The spaces P_ℓ

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

$W = \{\text{alcoves}\}$



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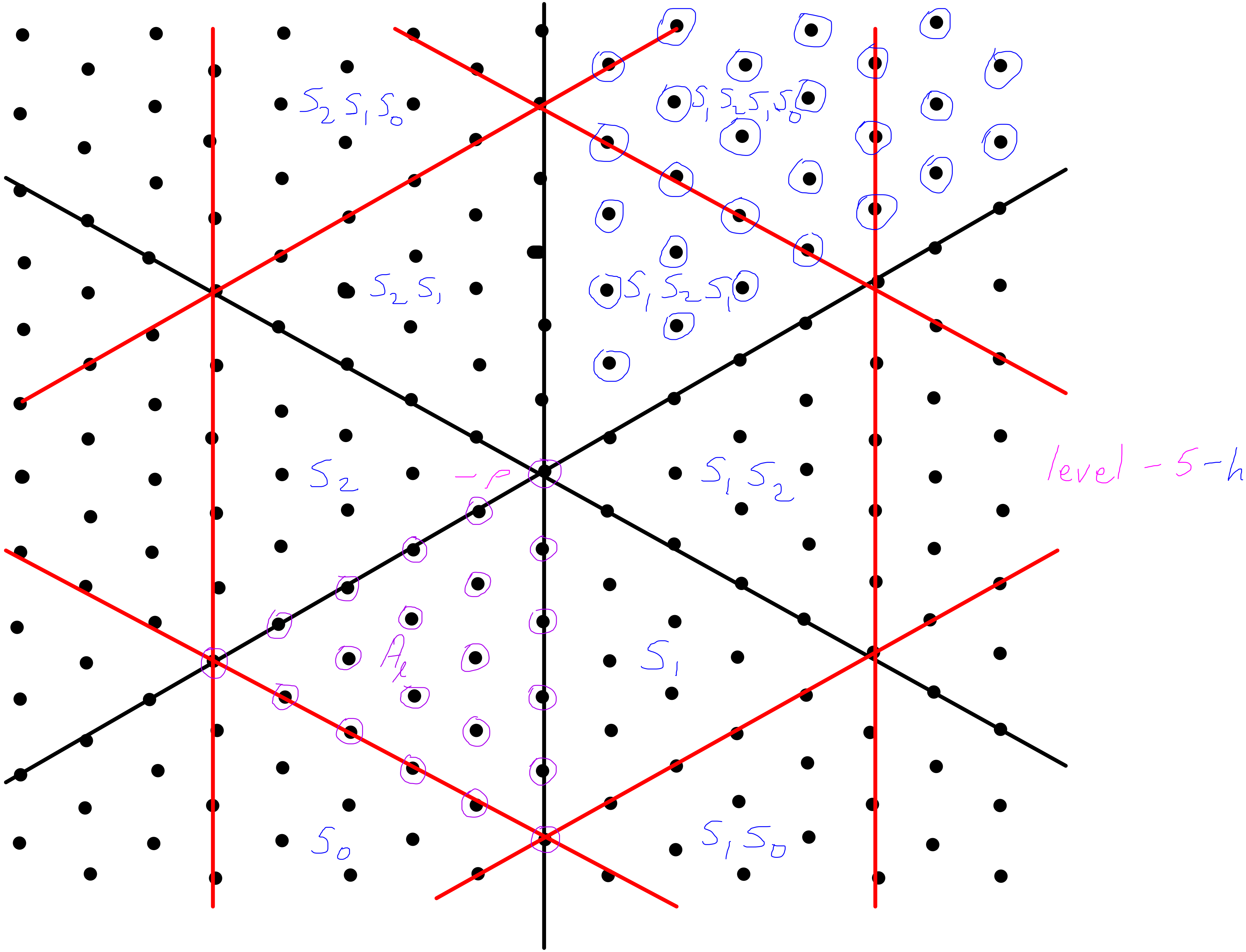
$s_1 s_0$

The spaces $\mathcal{P}_\ell$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

$$\mathcal{P} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell \quad \text{and} \quad \mathcal{P}_\ell = \bigoplus_{\nu \in A_\ell} H \mathbf{1}_\nu$$



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level $-5-h$
A_ℓ
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$s_1 s_0$

The spaces $\mathcal{P}_\ell$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

$$\mathcal{P} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell \quad \text{and} \quad \mathcal{P}_\ell = \bigoplus_{\nu \in A_\ell} H \mathbb{1}_\nu$$

with $T_w \mathbb{1}_\nu = (t^{\frac{1}{2}})^{\ell(w)} \mathbb{1}_\nu$, for $w \in W_\nu$

The spaces $\mathcal{P}_\ell$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

$$\mathcal{P} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell \quad \text{and} \quad \mathcal{P}_\ell = \bigoplus_{v \in A_\ell} H \mathbb{1}_v$$

with $T_w \mathbb{1}_v = (t^{\frac{1}{2}})^{\ell(w)} \mathbb{1}_v$, for $w \in W_v$

$\mathcal{P}_\ell$ has bases

$$\{T_\lambda \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}\} \quad \text{and} \quad \{X_\lambda \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}\}$$

The spaces $\mathcal{P}_\ell$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

$$\mathcal{P} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell \quad \text{and} \quad \mathcal{P}_\ell = \bigoplus_{v \in A_\ell} H \mathbb{1}_v$$

$\mathcal{P}_\ell$ has bases

$$\{T_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\} \quad \text{and} \quad \{X_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\}$$

Where

$$T_\lambda = T_{wv} = T_w \mathbb{1}_v \quad \text{and} \quad X_\lambda = X_{wv} = X^w \mathbb{1}_v$$

The spaces $\mathcal{P}_\ell$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

$$\mathcal{P} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell \quad \text{and} \quad \mathcal{P}_\ell = \bigoplus_{v \in A_\ell} H \mathbf{1}_v$$

$\mathcal{P}_\ell$ has bases $\{T_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\}$ and $\{X_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\}$

The spaces $\mathcal{P}_\ell$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

$$\mathcal{P} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell \quad \text{and} \quad \mathcal{P}_\ell = \bigoplus_{v \in A_\ell} H \mathbb{1}_v$$

$\mathcal{P}_\ell$ has bases $\{T_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\}$ and $\{X_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\}$

$\mathcal{P}_\ell$ has bar involution $\bar{\ } : \mathcal{P}_\ell \longrightarrow \mathcal{P}_\ell$

$$\overline{h \mathbb{1}_v} = \bar{h} \mathbb{1}_v, \quad \text{for } h \in H$$

The spaces $\mathcal{P}_\ell$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

$$\mathcal{P} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell \quad \text{and} \quad \mathcal{P}_\ell = \bigoplus_{v \in A_\ell} H \mathbb{1}_v$$

$\mathcal{P}_\ell$ has bases $\{T_\lambda \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}\}$ and $\{X_\lambda \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}\}$

$\mathcal{P}_\ell$ has bar involution $\overline{} : \mathcal{P}_\ell \longrightarrow \mathcal{P}_\ell$

$$\overline{h \mathbb{1}_v} = \bar{h} \mathbb{1}_v, \quad \text{for } h \in H$$

Let $\{C_\lambda \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}\}$ be the KL-basis of $\mathcal{P}_\ell$.

The spaces $\mathcal{P}_\ell$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

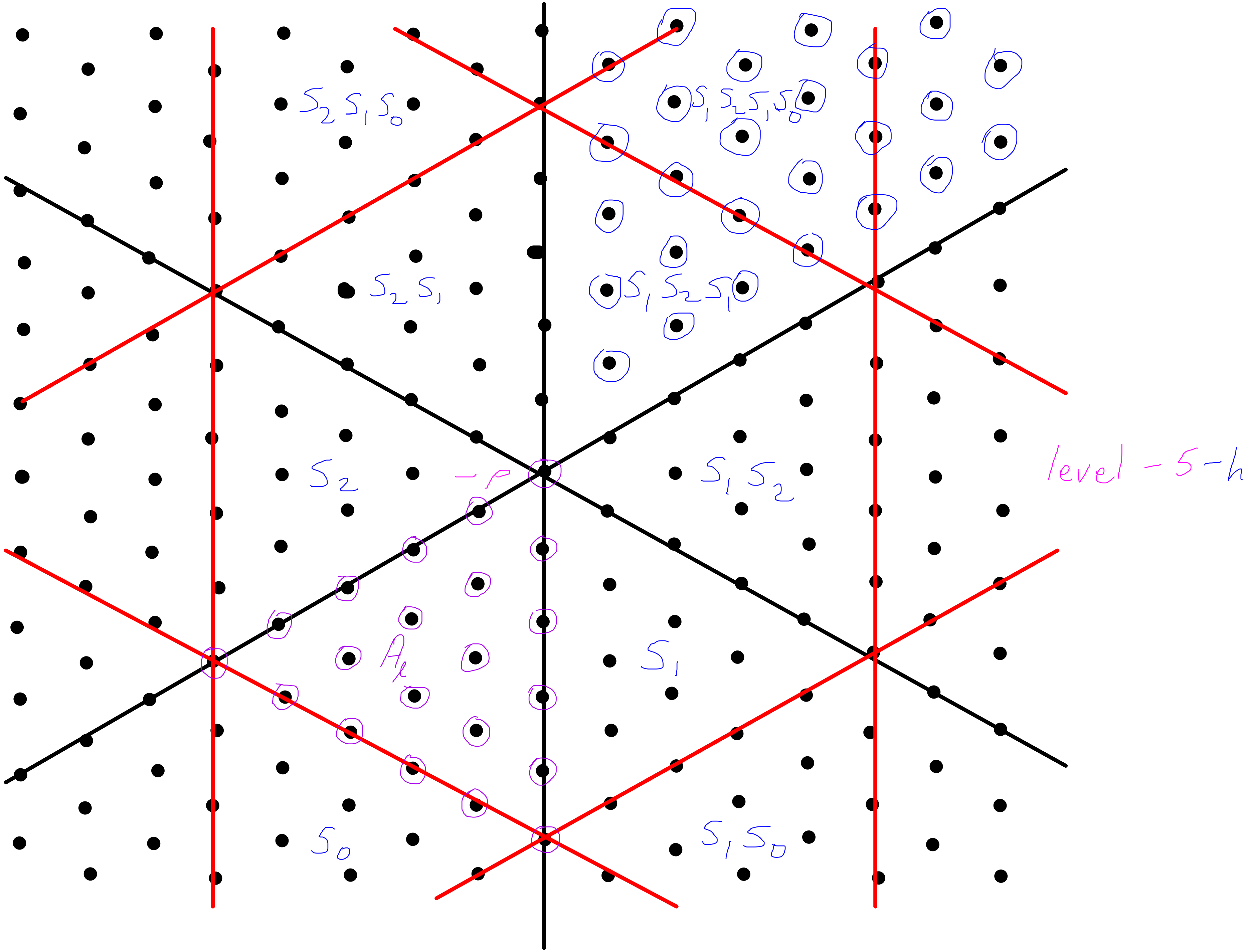
$$\mathcal{P} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell \quad \text{and} \quad \mathcal{P}_\ell = \bigoplus_{\nu \in A_\ell} H \mathbb{1}_\nu$$

$\mathcal{P}_\ell$ has bases $\{T_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\}$ and $\{C_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\}$

$$\text{Grothendieck group} \begin{pmatrix} \mathcal{O} \text{ for } \mathfrak{g} \\ \text{with level } \ell \end{pmatrix} \xrightarrow[\sim]{\text{Kashiwara-Tanisaki}} \mathcal{P}_\ell$$

$$[M(\lambda)] \longmapsto T_\lambda$$

$$[L(\lambda)] \longmapsto C_\lambda$$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A_ℓ
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The spaces $\mathcal{P}_\ell^+$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

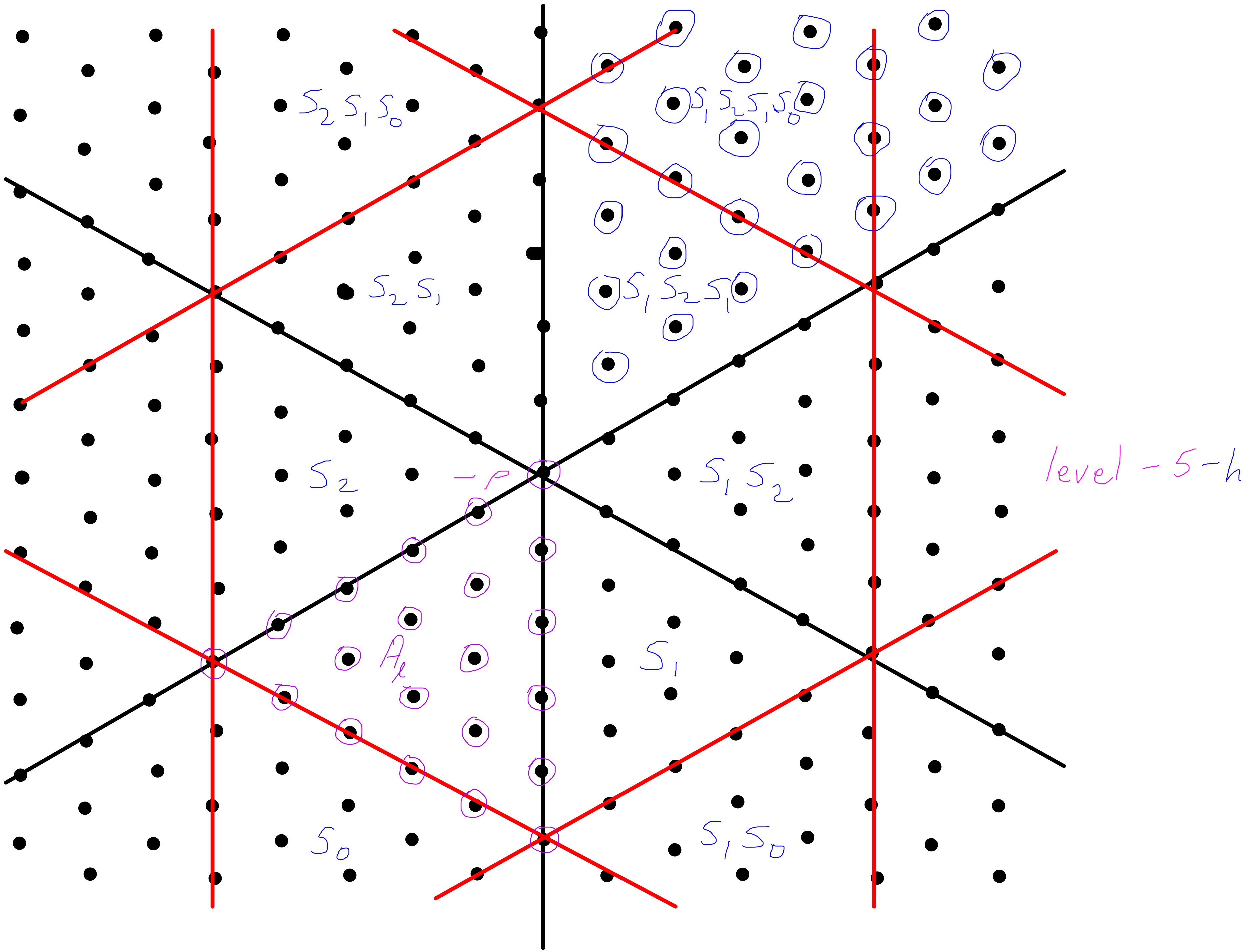
$$\mathcal{P}^+ = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell^+ \quad \text{and} \quad \mathcal{P}_\ell^+ = \bigoplus_{v \in A_\ell} \varepsilon_0 H \mathbb{1}_v$$

$\mathcal{P}_\ell$ has bases

$$\{[T_\lambda] \mid \lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+\} \quad \text{and} \quad \{[X_\lambda] \mid \lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+\}$$

Where

$$[T_\lambda] = [T_{wv}] = \varepsilon_0 T_w \mathbb{1}_v \quad \text{and} \quad [X_\lambda] = [X_{wv}] = \varepsilon_0 X^w \mathbb{1}_v$$



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level $-5-h$
A_ℓ
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The spaces $\mathcal{P}_\ell$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

$$\mathcal{P} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell \quad \text{and} \quad \mathcal{P}_\ell = \bigoplus_{v \in A_\ell} H \mathbb{1}_v$$

$\mathcal{P}_\ell$ has bases

$$\{T_\lambda \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}\} \quad \text{and} \quad \{X_\lambda \mid \lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}\}$$

Where

$$T_\lambda = T_{w_0 v} = T_w \mathbb{1}_v \quad \text{and} \quad X_\lambda = X_{w_0 v} = X^w \mathbb{1}_v$$

The spaces $\mathcal{P}_\ell^+$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

$$\mathcal{P}^+ = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell^+ \quad \text{and} \quad \mathcal{P}_\ell^+ = \bigoplus_{v \in A_\ell} \varepsilon_0 H \mathbb{1}_v$$

$\mathcal{P}_\ell$ has bases

$$\{[T_\lambda] \mid \lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+\} \quad \text{and} \quad \{[X_\lambda] \mid \lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+\}$$

where

$$[T_\lambda] = [T_{w_0 v}] = \varepsilon_0 T_w \mathbb{1}_v \quad \text{and} \quad [X_\lambda] = [X_{w_0 v}] = \varepsilon_0 X^w \mathbb{1}_v$$

and

$$\varepsilon_0 T_w = (-t^{-1/2})^{\ell(w)} \varepsilon_0, \quad \text{for } w \in W_0$$

The spaces $\mathcal{P}_\ell^+$

The affine Hecke algebra H has bases

$$\{T_w \mid w \in W\} \quad \text{and} \quad \{X^w \mid w \in W\}$$

$$\mathcal{P}^+ = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{P}_\ell^+ \quad \text{and} \quad \mathcal{P}_\ell^+ = \bigoplus_{\nu \in A_\ell} \varepsilon_0 H \mathbb{1}_\nu$$

$\mathcal{P}_\ell$ has bases $\{[X_\lambda] \mid \lambda \in (\mathfrak{h}_{\mathbb{Z}}^{*})^+\}$ and $\{[C_\lambda] \mid \lambda \in (\mathfrak{h}_{\mathbb{Z}}^{*})^+\}$

$$\text{Grothendieck group}\begin{pmatrix}\text{parabolic } \mathcal{O}^{\mathfrak{g}}_{\mathfrak{g}} \\ \text{with level } \ell\end{pmatrix} \xrightarrow[\sim]{\text{Kashiwara-Tanisaki}} \mathcal{P}_\ell^+$$

$$[\Delta(\lambda)] \longmapsto [X_\lambda]$$

$$[L(\lambda)] \longmapsto [C_\lambda]$$

The spaces $\mathcal{P}_\ell$ and $\mathcal{P}_\ell^+$

$$\text{Grothendieck group}\begin{pmatrix}\mathcal{O}\text{ for }\mathfrak{g}\\ \text{with level }\ell\end{pmatrix} \xrightarrow{\sim} \mathcal{P}_\ell$$

$$[M(\lambda)] \longmapsto T_\lambda$$

$$[L(\lambda)] \longmapsto C_\lambda$$

$$\text{Grothendieck group}\begin{pmatrix}\text{parabolic }\mathcal{O}^{\mathfrak{g}}_{\mathring{\mathfrak{g}}}\\ \text{with level }\ell\end{pmatrix} \xrightarrow{\sim} \mathcal{P}_\ell^+$$

$$[\Delta(\lambda)] \longmapsto [X_\lambda]$$

$$[L(\lambda)] \longmapsto [C_\lambda]$$

The spaces $\mathcal{P}_\ell$ and $\mathcal{P}_\ell^+$ and $\mathcal{F}_\ell$

$$\text{Grothendieck group}\begin{pmatrix}\mathcal{O} \text{ for } \mathfrak{g} \\ \text{with level } -\ell-h\end{pmatrix} \xrightarrow{\sim} \mathcal{P}_{-\ell-h}$$

$$[M(\lambda)] \longmapsto T_\lambda$$

$$[L(\lambda)] \longmapsto C_\lambda$$

$$\text{Grothendieck group}\begin{pmatrix}\text{parabolic } \mathcal{O}^{\mathfrak{g}}_{\mathfrak{g}} \\ \text{with level } -\ell-h\end{pmatrix} \xrightarrow{\sim} \mathcal{P}^+_{-\ell-h} \xrightarrow{\sim} \mathcal{F}_\ell$$

$$[\Delta(\lambda)] \longmapsto [X_\lambda] \longmapsto |\lambda\rangle$$

$$[L(\lambda)] \longmapsto [C_\lambda] \longmapsto C_\lambda$$

Let $\ell \in \mathbb{Z}_{>0}$. The level ℓ Fock space F_ℓ

has $\mathbb{Z}[q,q^{-1}]$-basis $\{ |\lambda\rangle \mid \lambda \in (\mathring{\mathfrak{h}}^*_{\mathbb{Z}})^+ \}$

$$\bar{q} = q^{-1} \quad \text{and} \quad \overline{|\lambda\rangle} = q^{\ell(w_\lambda)}(-q^{-1})^{\ell(w_0)} |w_0 \circ \lambda\rangle$$

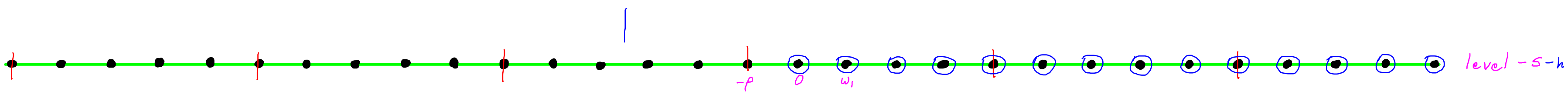
$$\overline{C_\lambda} = C_\lambda \quad \text{and} \quad C_\lambda = |\lambda\rangle + \sum_{\mu \neq \lambda} p_{\mu\lambda} |\mu\rangle$$

with $p_{\lambda\mu} \in q\mathbb{Z}[q]$.

$$\text{Grothendieck group}\begin{pmatrix}\text{finite dimensional} \\ U_\epsilon \mathring{\mathfrak{g}}\text{-modules} \\ \epsilon^\ell = 1\end{pmatrix} \xrightarrow{\sim} F_\ell$$

$$[\Delta_\epsilon(\lambda)] \longmapsto |\lambda\rangle$$

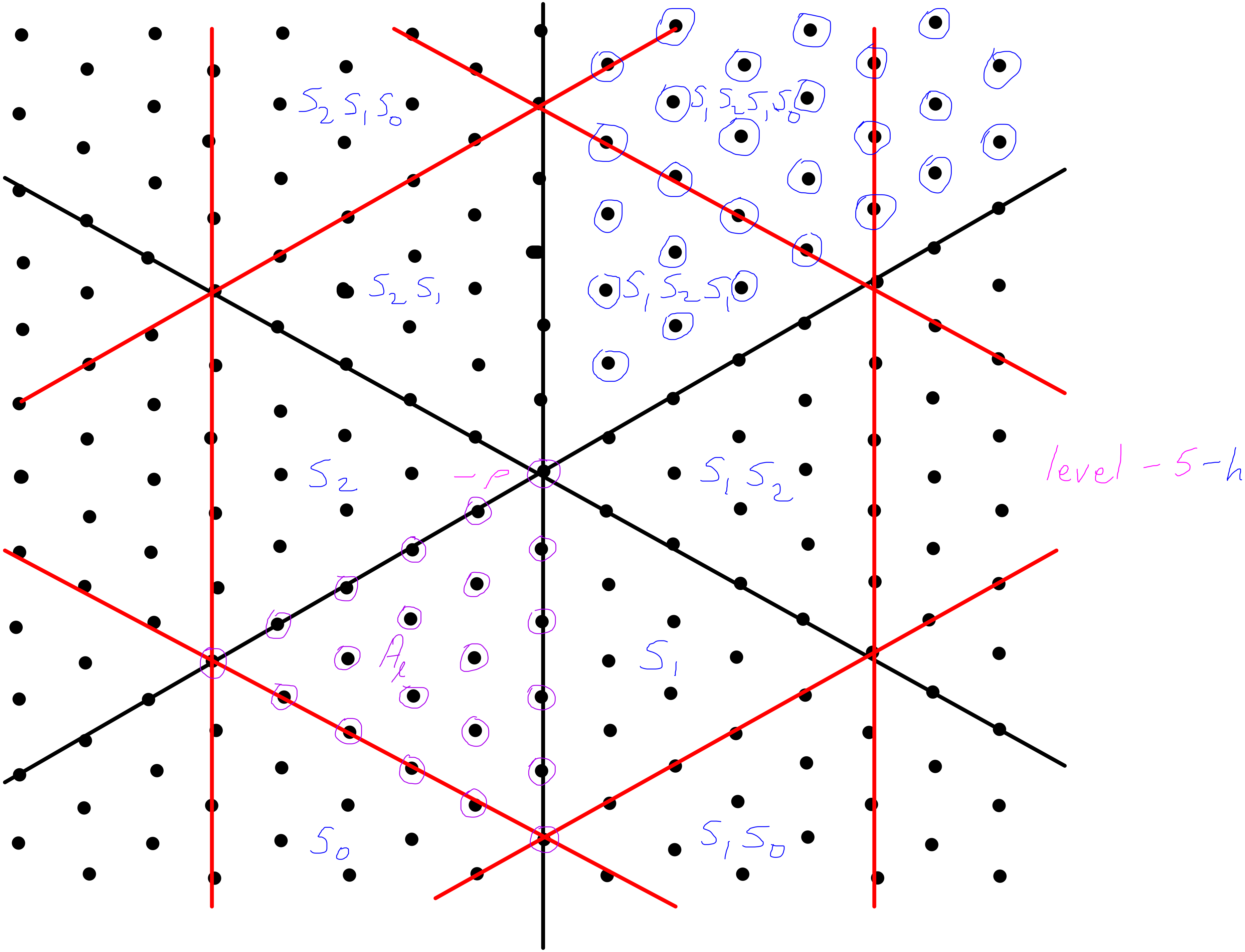
$$[L_\epsilon(\lambda)] \longmapsto C_\lambda$$



The spaces $\mathcal{P}_\ell$ and $\mathcal{P}_\ell^+$ and $\mathcal{F}_\ell$

$$
\begin{array}{ccc}
\text{Grothendieck group}\begin{pmatrix} \mathcal{O} \text{ for } \mathfrak{g} \\ \text{with level } \ell \end{pmatrix} & \xrightarrow{\sim} & \mathcal{P}_\ell \\
[M(\lambda)] & \longmapsto & T_\lambda \\
[L(\lambda)] & \longmapsto & C_\lambda
\end{array}
$$

$$
\begin{array}{ccccc}
\text{Grothendieck group}\begin{pmatrix} \text{parabolic } \mathcal{O}^{\mathfrak{g}}_{\mathfrak{g}} \\ \text{with level } \ell \end{pmatrix} & \xrightarrow{\sim} & \mathcal{P}_\ell^+ & \xrightarrow{\sim} & \mathcal{F}_\ell \\
[\Delta(\lambda)] & \longmapsto & [X_\lambda] & \longmapsto & |\lambda\rangle \\
[L(\lambda)] & \longmapsto & [C_\lambda] & \longmapsto & C_\lambda
\end{array}
$$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