# Affine Springer Fibers

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Goresky - Kottwitz - MacPherson, Rep. Theory 2006

$G$ complex reductive alg. group $\qquad \mathfrak{g} = Lie(G)$

$\cup|$ $\qquad \cup|$

$B$ Borel subgroup $\qquad \mathfrak{b} = Lie(B)$

$\cup|$ $\qquad \cup|$

$T$ maximal torus $\qquad \mathfrak{a} = Lie(T)$

Let

$$F = \mathbb{C}((\epsilon)), \qquad \begin{matrix} V \text{ a } G\text{-module}, \\ \cup\!\!\!| \\ v \end{matrix} \qquad \begin{matrix} y \in \mathfrak{a} \\ t \in \mathbb{R}. \end{matrix}$$

Then

$$\mathfrak{g}(F)_y = \Big(\bigoplus_{\alpha(y)+k \geq 0} \mathfrak{g}_\alpha \epsilon^k\Big) \oplus \Big(\bigoplus_{k \geq 0} \mathfrak{a}\epsilon^k\Big)$$

is the Lie algebra of a parahoric $G(F)_y$ in $G(F)$.

Let

$$V(F)_{y,\geq t} = \bigoplus_{\lambda(y)+m \geq t} V_\lambda \epsilon^m,$$

where $V_\lambda$ is the $\lambda$-weight space of $V$.

The <u>generalised affine Springer fiber</u> is

$$\mathcal{F}_y(t,v) = \left\{ g \in \frac{G(F)}{G(F)_y} \;\middle|\; g^{-1}v \in V(F)_{y,\geq t} \right\}$$

Special case:

$$y \in \text{int. of fund. alcove}, \quad V = \mathfrak{g}, \quad v = u \in \mathfrak{g}, \quad t = 0.$$

Then

$$G(F)_y = I, \quad \mathfrak{g}_y = \mathrm{Lie}(I), \quad V(F)_{y,\geq 0} = \mathfrak{g}(F)_{y,\geq 0} = \mathrm{Lie}(I)$$

where

$$\mathrm{Lie}(I) = \Big(\bigoplus_{\substack{\alpha \in R \\ k \in \mathbb{Z}_{>0}}} \mathfrak{g}_\alpha t^k\Big) \oplus \Big(\bigoplus_{\substack{\alpha \in R^+ \\ k \in \mathbb{Z}_{\geq 0}}} \mathfrak{g}_{-\alpha} t^k\Big) \oplus \Big(\bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{h} t^k\Big)$$

The affine Springer fiber of [Kazhdan-Lusztig 1988] is

$$\mathcal{F}_y(u) = \mathcal{F}_y(t,v) = \{ gI \in G(F)/I \mid \mathrm{Ad}_{g^{-1}}(u) \in \mathrm{Lie}(I) \}.$$

Chopping up $\mathcal{F}_y(t,v)$

Let

$x \in \mathfrak{a}$, $s \in \mathbb{R}$ with
(a) $s \geq t$
(b) $v \in V(F)_{x,\geq s}$
(c) $\bar{v}$ is a G-good vector in $V$

where

$$\begin{array}{ccccc} V(F)_{x,\geq s} & \twoheadrightarrow & V(F)_{x,=s} & \longrightarrow & V(x, s+\mathbb{Z}) \subseteq V \\ v & \longmapsto & v_s & \longmapsto & \bar{v} \end{array}$$

Define

$$S = \mathcal{F}_y(t,v) \cap G(F)_x \cdot G(F)_y,$$

the intersection of $\mathcal{F}_y(t,v)$ with the $G(F)_x$-orbit of $G(F)_y$ in $G(F)/G(F)_y$.

Example (see [Oblomkov–Yun Theorem 5.4.2 (2)])

$$Sp_\gamma = \mathcal{F}_\gamma(0,\gamma) = \coprod_{w \in W} (Sp_\gamma \cap IwI)$$

[GKM, §4.3] say $\mathcal{F}_y(t,v) = \coprod_{G(F)_x \backslash G / G(F)_y} (\mathcal{F}_y(t,v) \cap G(F)_x w G(F)_y)$

and then note that the analysis of ([GKM §4.3 paragraph 2])

$\mathcal{F}_y(t,v) \cap G(F)_x w G(F)_y$ ~~is equivalent to~~

$\simeq \mathcal{F}_{wy}(t,v) \cap G(F)_x \cdot G(F)_y$.

Structure of $S$

$$S = S_{r_n} = S_{r_{n-1}+}$$

$\downarrow$ affine space bundle

$\vdots$

$\downarrow$ affine space bundle

$S_{r_3} = S_{r_2+}$

$\downarrow$ affine space bundle

$S_{r_2} = S_{r_1+}$

$\downarrow$ affine space bundle

$S_{r_1} = S_{0+} = \mathcal{P}_{y-x}(t-s, \bar{v})$ a Hessenberg variety

with $r_n > r_{n-1} > \dots > r_3 > r_2 > r_1 > 0$

where

$$\begin{array}{ccc} \check{S}_{r+} & & S_{r+} = G(F)_{x,\geq r} \backslash \check{S}_{r+} \\ \cap\mid & \text{gives} & \downarrow \\ \check{S}_r & & S_r = G(F)_{x,\geq r} \backslash \check{S}_r \end{array}$$

with

$$\check{S}_{r+} = \left\{ g \in \frac{G(F)_x}{G(F)_x \cap G(F)_y} \;\middle|\; g^{-1}v \in \left( V(F)_{y,\geq t} + V(F)_{x,>s+r} \right) \right\}$$

$$\cap\mid$$

$$\check{S}_r = \left\{ g \in \frac{G(F)_x}{G(F)_x \cap G(F)_y} \;\middle|\; g^{-1}v \in \left( V(F)_{y,\geq t} + V(F)_{x,\geq s+r} \right) \right\}$$

## Hessenberg varieties

$$\mathfrak{p}_y = \bigoplus_{\alpha(y)\geq 0} \mathfrak{g}_\alpha$$

is the Lie algebra of a parabolic $P_y$ in $G$. Let

$$V_{y,\geq t} = \bigoplus_{\lambda(y)\geq t} V_\lambda .$$

A Hessenberg variety is (see [GKM, §2.5])

$$\mathcal{P}_y(t,v) = \{ g \in G/P_y \mid g^{-1}v \in V_{y,\geq t} \}$$

$$= \{ gP_y \mid v \in gP_y V_{y,\geq t} \} \subseteq G/P_y .$$

for $v \in V$ a $G$-good vector ($G\cdot v$ is "big") and $t \in \mathbb{R}_{\leq 0}$.

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# Matching $S_{0t}$ and $P_{y-x}(t-s,\bar{v})$.

Let

$$H_0 = G_x(\mathcal{O}) = G(F)_x(\mathcal{O}) \quad \text{with} \quad \mathrm{Lie}(H_0) = \bigoplus_{\alpha(x)+k \geq 0} \mathfrak{g}_\alpha \epsilon^k$$

$$\cup\!|$$

$$P = F_y^0 G_x(\mathcal{O}) \quad \text{with} \quad \mathrm{Lie}(P) = \bigoplus_{\substack{\alpha(x)+k \geq 0 \\ \alpha(y) \geq 0}} \mathfrak{g}_\alpha \epsilon^k$$

Let $H = A \cdot \prod_{\substack{\alpha \in R \\ \alpha(x) \in \mathbb{Z}}} \mathcal{X}_\alpha$ with Lie algebra $\bigoplus_{\alpha(x) \in \mathbb{Z}} \mathfrak{g}_\alpha$.

Then $P_{y-x}$ has Lie algebra $\bigoplus_{\alpha(y) \geq 0} \mathfrak{g}_\alpha$

$$\begin{tikzcd} H_0 \arrow[r, "\sim"] & H \\ \cup\!| & \cup\!| \\ P \arrow[r] & P_{y-x} \end{tikzcd}$$

# Affine Springer Fibers; Goresky-Kottwitz-MacPherson  ①
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Let

$F = \mathbb{C}((\epsilon))$, $\quad V$ a $G$-module, $\quad y \in \mathfrak{a}$,
$\quad v \in V$, $\quad t \in \mathbb{R}$.

Then

$$\mathfrak{g}(F)_y = \mathfrak{a}(F) \oplus \Big( \bigoplus_{\alpha(y)+k\geq 0} \mathfrak{g}_\alpha \epsilon^k \Big)$$

is the Lie algebra of a parahoric $G(F)_y$ in $G(F)$.

$$V(F)_{y,\geq t} = \bigoplus_{\lambda(y)+m\geq t} V_\lambda \epsilon^m$$

where $V_\lambda$ is the $\lambda$-weight space of $V$. The <u>generalized affine Springer fiber</u> is

$$\mathcal{F}_y(t,v) = \Big\{ g \in \frac{G(F)}{G(F)_y} \ \Big|\ g^{-1}v \in V(F)_{y,\geq t} \Big\}$$

$$= \{ gG(F)_y \mid v \in gG(F)_y V(F)_{y,\geq t} \}$$

If $y \in$ int. of an alcove, $V = \mathfrak{g}$, $v = u \in \mathfrak{g}$ and $t = 0$ then

$$G(F)_y = I, \quad \mathfrak{g}_y = \mathrm{Lie}(I), \quad V(F)_{y,\geq 0} = \mathfrak{g}(F)_{y,\geq 0} = \mathrm{Lie}(I)$$

and

$$\mathcal{F}_y(t,v) = \{ gI \mid \mathrm{Ad}_{g^{-1}}(u) \in \mathrm{Lie}(I) \} = \mathcal{F}_y(u)$$

the <u>affine Springer fiber</u> of [KL 88].

## Hessenberg varieties

$$\mathfrak{p}_y = \bigoplus_{\alpha(y)\geq 0} \mathfrak{g}_\alpha$$

is the Lie algebra of a parabolic $P_y$ in $G$.

$$V_{y,\geq t} = \bigoplus_{\lambda(y)\geq t} V_\lambda$$

A Hessenberg variety is (see §2.5)

$$P_y(t,v) = \{ g \in G/P_y \mid g^{-1}v \in V_{y,\geq t} \}$$
$$= \{ gP_y \mid v \in gP_y V_{y,\geq t} \} \subseteq G/P_y.$$

for $v \in V$ a $G$-good vector ($G\cdot v$ is "big")
$t \in \mathbb{R}_{\leq 0}$.

Theorem ([GKM §2.6]) Recalling that $H^*(G/B) = \frac{S(\mathfrak{a}^*)}{I}$.

$$P_y(t,v) = \phi \iff \prod_{\substack{\lambda \in \mathfrak{a}^* \\ (V/V_{y,\geq t})_\lambda \neq 0}} \lambda^{\dim (V/V_{y,\geq t})_\lambda} = 0 \text{ in } H^*(G/B)$$

# Intersections: $\mathcal{F}_y(t,v)$ and a $G(F)_x$-orbit.

Let

$x \in \mathfrak{a}$ with (a) $s \geq t$

$s \in \mathbb{R}$ (b) $v \in V(F)_{x, \geq s}$

(c)

Define

$$S = \mathcal{F}_y(t,v) \cap G(F)_x \cdot G(F)_y$$

$$= \left\{ g \in \frac{G(F)_x}{G(F)_x \cap G(F)_y} \;\middle|\; v \in g(G(F)_x \cap G(F)_y) V(F)_{y, \geq t} \right\}$$

$$= \left\{ g \in \frac{G(F)_x}{G(F)_x \cap G(F)_y} \;\middle|\; \sigma(g) = 0 \right\}$$

where

$$\begin{tikzcd} V \arrow[d, "\pi"'] \\ \dfrac{G(F)_x}{G(F)_x \cap G(F)_y} \arrow[u, bend right, "\sigma"'] \end{tikzcd} \qquad \begin{array}{c} \sigma(g) = v \in \dfrac{V(F)_{x, \geq s}}{V(F)_{x, \geq s} \cap g V(F)_{y, \geq t}} \\ \uparrow \\ g \end{array}$$

Note:

$$\dim\left( \frac{\mathfrak{g}(F)_x}{\mathfrak{g}(F)_x \cap \mathfrak{g}(F)_y} \right) = \mathrm{Card}\left\{ \begin{matrix}\text{roots} \\ \alpha + k\delta\end{matrix} \;\middle|\; \begin{matrix} 0 \leq \alpha(x) + k \\ \alpha(y) + k < 0 \end{matrix} \right\}$$

$$= \mathrm{Card}\{ \alpha + k\delta \mid \mathfrak{h}^{\alpha + k\delta} \text{ is between } x \text{ and } y \}.$$

## Dimension of $S$

$$\dim(S) = \dim\left(\frac{\mathfrak{g}(F)_x}{\mathfrak{g}(F)_x \cap \mathfrak{g}(F)_y}\right) - \dim\left(\frac{V(F)_{x,\geq s}}{V(F)_{x,\geq s} \cap V(F)_{y,\geq t}}\right)$$

Special case: $V(F) = \mathfrak{g}(F)$ and $t = 0$

$$\dim(S) = \left\{ \alpha + k\delta \;\middle|\; \begin{array}{l} 0 \leq \alpha(x) + k < s \\ \alpha(y) + k < 0 \end{array} \right\}$$

$$= \mathrm{Card}\left\{ \mathfrak{h}^{\alpha+k\delta} \;\middle|\; \begin{array}{l} \mathfrak{h}^{\alpha+k\delta} \text{ is between } x \text{ and } y \\ x \text{ is } \overset{\text{less than}}{\cancel{\text{at most}}} \text{ distance } s \text{ from } \mathfrak{h}^{\alpha+k\delta} \end{array} \right\}$$

## Structure of $S$

$$\begin{array}{l}
S = S_{r_n} = S_{r_{n-1}+} \\
\quad \downarrow \text{ affine space bundle} \\
\quad \vdots \\
\quad \downarrow \text{ affine space bundle} \\
S_{r_3} = S_{r_2+} \\
\quad \downarrow \text{ affine space bundle} \\
S_{r_2} = S_{r_1+} \\
\quad \downarrow \text{ affine space bundle} \\
S_{r_1} = S_{0,+} = \text{a Hessenberg } P_{y-x}(t-s, \bar{V})
\end{array}$$

GKM reading 10.06.2016 A. Roe (5)

<u>The slices</u> Let $x \in R$ and $s \in \mathbb{R}$ such that (a) $s \geq t$ (b) $v \in V(F)_{x,\geq s}$ (c)

~~For $x \in \mathcal{B}/R$ that~~

$$S = \cancel{G(F)_x \cdot G(F)_y \cap \mathcal{B}_y(t,v)}\ \mathcal{F}_y(t,v) \cap G(F)_x \cdot G(F)_y$$

the intersection of $\mathcal{F}_y(t,v)$ and the $G(F)_x$ orbit of $G(F_y)$.

$$= \left\{ g \in \frac{G(F)_x}{G(F)_x \cap G(F)_y} \middle| v \in g(G(F)_x \cap G(F)_y) V(F)_{y,\geq t} \right\}$$

Then

$$\begin{array}{lll}
0 & & \\
\wedge & & \\
r_1 & \check{S}_{r_1} = S_{0+\delta_0} & S_{0+} = S_{r_1} = G_{x,r_1} \backslash \check{S}_{r_1} \\
\wedge & \uparrow & \uparrow \text{affine space bundle} \\
r_2 & \check{S}_{r_2} = S_{r_1+\delta_1} & S_{r_1+\delta_1} = S_{r_2} = G_{x,r_2} \backslash \check{S}_{r_2} \\
\wedge & \uparrow & \uparrow \text{affine space bundle} \\
r_3 & \check{S}_{r_3} = S_{r_2+\delta_2} \quad \text{and taking } G_{x,r}\text{-orbits} & S_{r_2+\delta_2} = S_{r_3} = G_{x,r_3} \backslash \check{S}_{r_3} \\
\wedge & \uparrow & \uparrow \text{affine space bundle} \\
\vdots & \vdots & \vdots \\
 & \uparrow & \uparrow \\
 & \check{S} & S
\end{array}$$

$$\begin{array}{ccc}
\varphi_r^*(E_r) & \longrightarrow & E_r \\
\downarrow & & \downarrow \\
S_r & \xrightarrow{\varphi_r} & S_{0+}
\end{array}
\qquad
\begin{array}{c}
E_0 = T(S_{0+}) \\
\downarrow \\
S_{0+}
\end{array}$$

So

$$\dim(S) = \sum_{r \geq 0} \dim(E_r) \quad \text{with } \dim(E_r) = 0 \text{ for } r \gg 0.$$

$$\hat{S}_{r+} = \left\{ g \in \frac{G(F)_x}{G(F)_x \cap G(F)_y} \;\middle|\; g^{-1}v \in (V(F)_{y,\geq t} + V(F)_{x,>s+r}) \right\}$$

$\cap|$

$$\hat{S}_r = \left\{ g \in \frac{G(F)_x}{G(F)_x \cap G(F)_y} \;\middle|\; g^{-1}v \in (V(F)_{y,\geq t} + V(F)_{x,\geq s+r}) \right\}$$

and

$$\begin{array}{c} S_{r+} = G(F)_{x,>r} \setminus \hat{S}_{r+} \\ \downarrow \\ S_r = G(F)_{x,\geq r} \setminus \hat{S}_r \end{array}$$

Then

$$S_{0+} = \left\{ g \in \frac{G(F)_{x,=0}}{P} \;\middle|\; g^{-1}v_s \in F_y^{\,t} V(F)_{x,=s} \right\}$$

$$= \mathcal{P}_{y-x}(t-s, \bar{v})$$

is a Hessenberg variety.

$$V(F)_{x,\geq s} \to V(F)_{x,=s} \to 0$$

$$v \longmapsto v_s$$

$\cancel{V_{x,=s} \to V}$

$$V(F)_{x,=s} \xrightarrow{\sim} V(x,\underline{s}) = V(x, s+\mathbb{Z})$$

$$v_s \longmapsto \bar{v}.$$