Double affine Hecke algebra $\tilde{H}$

Ram-Yip, arXiv:0803.1146

$\tilde{H}$ is given by generators

$T_0, T_1, \dots, T_n$, $T_0^\vee, T_1^\vee, \dots, T_n^\vee$, $t_0, t_1, \dots, t_n, t_0^\vee, t_1^\vee, \dots, t_n^\vee$

$X^{\omega_1}, \dots, X^{\omega_n}, Y^{\alpha_1^\vee}, \dots, Y^{\alpha_n^\vee}, g \in \Omega$

and relations .....

$\tilde{H}$ is an algebra over $\mathbb{C}[q^{\pm 1}, t^{\pm \frac{1}{2}}] = \mathbb{C}_{q,t}$

(or its field of fractions or some finite extension of it).

Basis:

$$\tilde{H} = \mathbb{C}_{q,t}\text{-span}\left\{ X^\mu T_w Y^\lambda \;\middle|\; \begin{array}{c} \mu \in \mathfrak{a}_{\mathbb{Z}}^*, \; \lambda \in \mathfrak{a}_{\mathbb{Z}} \\ w \in W_0 \end{array} \right\}$$

W_0 is a finite reflection group $W_0 \subseteq GL_n(\mathfrak{a}_{\mathbb{Z}}^*)$.

$\mathfrak{a}_{\mathbb{Z}}^*$ is a free $\mathbb{Z}$-module; $\mathfrak{a}_{\mathbb{Z}}^* = \text{span}\{\omega_1, \dots, \omega_n\}$

$\mathfrak{a}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module; $\mathfrak{a}_{\mathbb{Z}} = \text{span}\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$

$\langle\ \rangle : \mathfrak{a}_{\mathbb{Z}}^* \times \mathfrak{a}_{\mathbb{Z}} \to \mathbb{Z}$ with $\langle w\mu, w\lambda\rangle = \langle \mu, \lambda\rangle$.

The favorite example is $W_0 = S_n$ acting on

$\mathfrak{a}_{\mathbb{Z}}^* = \text{span}\{\varepsilon_1, \dots, \varepsilon_n\}$ and $\mathfrak{a}_{\mathbb{Z}} = \text{span}\{\varepsilon_1^\vee, \dots, \varepsilon_n^\vee\}$

by permuting $\varepsilon_1, \dots, \varepsilon_n$ and permuting $\varepsilon_1^\vee, \dots, \varepsilon_n^\vee$

with $\langle \varepsilon_i^\vee, \varepsilon_j \rangle = \delta_{ij}$.

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<u>Subalgebras of $\tilde{H}$</u>

(a) $\mathbb{C}[Y] = \mathbb{C}_{q,t}\text{-span}\{ Y^\lambda \mid \lambda \in \mathfrak{a}_\mathbb{Z} \}$

$= \mathbb{C}_{q,t}[Y^{\alpha_1^\vee}, \ldots, Y^{\alpha_n^\vee}]$ with

$Y^\lambda = (Y^{\alpha_1^\vee})^{\lambda_1} \cdots (Y^{\alpha_n^\vee})^{\lambda_n}$ for $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$

$Y^\lambda Y^\gamma = Y^{\lambda+\gamma}$

(b) $\mathbb{C}[X] = \mathbb{C}_{q,t}\text{-span}\{ X^\mu \mid \mu \in \mathfrak{a}^*_\mathbb{Z} \}$

$= \mathbb{C}_{q,t}[X^{\omega_1}, \ldots, X^{\omega_n}]$ with

$X^\mu = (X^{\omega_1})^{\mu_1} \cdots (X^{\omega_n})^{\mu_n}$ for $\mu_1, \ldots, \mu_n \in \mathbb{Z}^n$

$X^\mu X^\nu = X^{\mu+\nu}$

(c) $H_0 = \mathbb{C}_{q,t}\text{-span}\{ T_w \mid w \in W_0 \}$ is ~~given by~~ a "finite Hecke algebra", given by generators: $T_1, \ldots, T_n$

relations: $\underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}}$

$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_i + 1$

(d) $H_X = \mathbb{C}_{q,t}\text{-span}\{ X^\mu T_w \mid \mu \in \mathfrak{a}^*_\mathbb{Z}, w \in W_0 \}$

$H_Y = \mathbb{C}_{q,t}\text{-span}\{ T_w Y^\lambda \mid \lambda \in \mathfrak{a}^*_\mathbb{Z}, w \in W_0 \}$

are <u>affine Hecke algebras</u>.

H_Y is generated by $T_0, T_1, \ldots, T_n$.

The polynomial representation

Recall

$$\tilde{H} = \mathbb{C}_{q,t}\text{-span}\left\{ X^\mu T_w Y^\lambda \;\middle|\; \begin{matrix}\mu \in \mathfrak{a}^*_\mathbb{Z}, \lambda \in \mathfrak{a}_\mathbb{Z} \\ w \in W_0\end{matrix} \right\}$$

The polynomial representation is

$$\tilde{H}\mathbb{1} = \mathbb{C}_{q,t}\text{-span}\{ X^\mu \mathbb{1} \mid \mu \in \mathfrak{a}^*_\mathbb{Z} \} = \mathbb{C}[X]\cdot\mathbb{1}.$$

with $T_i \mathbb{1} = t^{\frac{1}{2}}\mathbb{1}$ for $i = 1, \ldots, n$

$Y^\lambda \mathbb{1} = q^{\langle \lambda^\vee, \rho_c \rangle}\mathbb{1}$

(alternatively $T_0 \mathbb{1} = t^{\frac{1}{2}}\mathbb{1}$ and $t = q^c$, c is a "level").

The Demazure-Lusztig operators are the action of T_i:

$$T_i X^\mu \mathbb{1} = \left(t^{\frac{1}{2}} X^{s_i \mu} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{X^\mu - X^{s_i \mu}}{1 - X^{\alpha_i}} \right)\mathbb{1}, \text{ for } i = 0, 1, \ldots, n.$$

The Cherednik-Dunkl operators are the action of Y^λ.

The nonsymmetric Macdonald polynomials are given by

$E_0 = \mathbb{1}$, and $E_{s_i w} = \tau_i^\vee E_w$, for $i = 0, 1, \ldots, n$, $s_i w > w$

where $\tau_i^\vee = T_i + \dfrac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - Y^{-\alpha_i}}$

They are the eigenvectors of the Cherednik-Dunkl operators.

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DAHA-Jones polynomials (for torus knots)

The symmetric Macdonald polynomials are

$$P_\mu = \mathbb{1}_0 E_\mu$$

where $\mathbb{1}_0 = \sum_{w \in W_0} (t^{-\frac{1}{2}})^{l(w_0 w)} T_w$, more importantly

$\mathbb{1}_0 T_i = t^{\frac{1}{2}} \mathbb{1}_0$ for $i = 1, 2, \ldots, n$.

The group $PSL_2^\wedge(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$

acts on $\hat{H}$ (and on $\mathbb{1}_0 \hat{H} \mathbb{1}$) by automorphisms:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(X^\lambda) = (X^\lambda)^a (Y^\lambda)^c, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}(T_i) = T_i.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(Y^\mu) = (X^\mu)^b (Y^\mu)^d,$$

Define ~~ev~~

$$ps: \mathbb{C}_{q,t}[X] \longrightarrow \mathbb{C}_{q,t}$$
$$X^\mu \longmapsto q^{\langle \mu, \rho_c \rangle}$$

$$ev: \hat{H} \longrightarrow \mathbb{C}_{q,t}$$
$$X^\mu \longmapsto q^{-\langle \mu, \rho_c \rangle}$$
$$T_i \longmapsto t^{\frac{1}{2}}$$
$$Y^\lambda \longmapsto q^{\langle \lambda, \rho_c \rangle}$$

Let $\gamma = \begin{pmatrix} r & * \\ s & * \end{pmatrix}$ in $PSL_2^\wedge(\mathbb{Z})$. The DAHA-Jones polynomial is

$$JD^R_{r,s}(\mu; q, t) = \frac{1}{ps(P_\mu)} \cdot ev(\gamma(P_\mu)) = ev\left(\gamma\left(\frac{P_\mu}{ps(P_\mu)}\right)\right).$$

Elliot-Gukov, arXiv 1505.01635

Geometric representations

Let $G = G(\mathbb{C}((\varepsilon)))$ and I an Iwahori subgroup.

$$\begin{array}{ccc}
G = G(\mathbb{C}((\varepsilon))) & & \\
\cup\!| & & \\
K = G(\mathbb{C}[[\varepsilon]]) & \xrightarrow[\varepsilon=0]{\Phi} & G(\mathbb{C}) \\
\cup\!| & & \cup\!| \\
I = \Phi^{-1}(B) & \longrightarrow & B = \{\text{upper triangular}\}
\end{array}$$

G/I is the affine flag variety.

$\cup\!|$

Sp_a affine Springer fibers.

$K_{\mathbb{C}_m}^{dil}(G/I)$ is the polynomial representation of $\tilde{H}$

$K_{\mathbb{C}_m}^{dil}(Sp_a)$ are other representations of $\tilde{H}$.

Conversion to $H\mathfrak{J}^{r}$

Replace all q, $t^{\frac{1}{2}}$, y^{λ} and T_i

with r, c, y_λ and t_{s_i}

by using the formulas

$$q = e^{r}, \quad t^{\frac{1}{2}} = e^{rc/2}, \quad y^{\lambda} = e^{r y_\lambda}$$

$$T_i - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - y^{-\alpha_i^{\vee}}} = t_{s_i} - \frac{c}{y_{\alpha_i^{\vee}}}$$

The eigenvectors of the y_λ are the Jack polynomials.