"A combinatorial gadget for decomposition numbers for quantum groups at roots of unity" ①

CUNY Representation Theory Seminar Thursday 29.04.2016

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Combinatorial Fock space

$$\mathcal{F}_\ell = \mathbb{Z}[q,q^{-1}]\text{-span}\{|\lambda\rangle \mid \lambda \in \mathfrak{a}^*_{\mathbb{Z}}\}$$

with

$$|s_i \circ \lambda\rangle = \begin{cases} -|\lambda\rangle, & \text{if } \langle \lambda+\rho, \alpha_i^\vee\rangle \in \ell\mathbb{Z}_{\geq 0}, \\ -q|\lambda\rangle, & \text{if } 0 < \langle \lambda+\rho, \alpha_i^\vee\rangle < \ell, \\ -q|s_i\circ\lambda^{(1)}\rangle - |\lambda^{(1)}\rangle - q|\lambda\rangle, & \text{if } \langle \lambda+\rho, \alpha_i^\vee\rangle > \ell \text{ and } \notin \ell\mathbb{Z}_{\geq 0}. \end{cases}$$

Define a $\mathbb{Z}$-linear involution $\bar{\ }: \mathcal{F}_\ell \to \mathcal{F}_\ell$ by

$$\bar{q} = q^{-1} \quad \text{and} \quad \overline{|\lambda\rangle} = q^{\ell(w_\lambda)}(-q^{-1})^{\ell(w_0)}|w_0\circ\lambda\rangle.$$

$\mathcal{F}_\ell$ has two bases

$$\{|\lambda\rangle \mid \lambda \in (\mathfrak{a}^*_{\mathbb{Z}})^+\} \quad \text{and} \quad \{C_\lambda \mid \lambda \in (\mathfrak{a}^*_{\mathbb{Z}})^+\}$$

where C_λ is determined by

$$\overline{C_\lambda} = C_\lambda \quad \text{and} \quad C_\lambda = |\lambda\rangle + \sum_\mu p_{\lambda\mu}|\mu\rangle$$

with $p_{\lambda\mu} \in q\mathbb{Z}[q]$.

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<u>The theorem</u> $\varepsilon = e^{2\pi i/\ell}$

$$F_\ell \xrightarrow{\sim} \text{Groth}\begin{pmatrix}\text{fin dim'l} \\ \text{group} \; U_\varepsilon\mathfrak{g}\text{-modules}\end{pmatrix}$$

$$|\lambda\rangle \longmapsto \Delta_\varepsilon(\lambda) \quad \text{Weyl module}$$

$$C_\lambda \longmapsto L_\varepsilon(\lambda) \quad \text{Simple module}$$

If $\Delta_\varepsilon(\lambda) = \Delta_\varepsilon(\lambda)^{(0)} \supseteq \Delta_\varepsilon(\lambda)^{(1)} \supseteq \cdots$ is the Jantzen filtration

then

$$|\lambda\rangle = \sum_\mu Q_{\lambda\mu}(q) C_\mu \quad \text{with} \quad Q_{\lambda\mu}(q) = \sum_{j\in\mathbb{Z}} q^j \left[\frac{\Delta_\varepsilon(\lambda)^{(j)}}{\Delta_\varepsilon(\lambda)^{(j+1)}} : L_\varepsilon(\mu)\right]$$

<u>The proof</u>

$$F_\ell \longrightarrow \bigoplus_{\nu\in\mathfrak{h}_{-\ell-h}} \varepsilon_0 H \mathbb{1}_\nu \longrightarrow \text{Groth group}\begin{pmatrix}\text{integral weight} \\ \hat{\mathfrak{g}}\text{-modules} \\ \text{in } \mathcal{O}^{\hat{\mathfrak{g}}} \text{ of level} \\ -\ell-h\end{pmatrix} \longrightarrow \text{Groth group}\begin{pmatrix}\text{fin.dim'l} \\ U_\varepsilon\mathfrak{g}\text{-modules}\end{pmatrix}$$

$$|\lambda\rangle \longmapsto [X_\lambda] \longmapsto \Delta^{\hat{\mathfrak{g}}}_q(\lambda) \longmapsto \Delta_\varepsilon(\lambda)$$

$$C_\lambda \longmapsto [C_\lambda] \longmapsto L(\lambda) \longmapsto L_\varepsilon(\lambda).$$

$$q \longmapsto t^{\frac{1}{2}}$$

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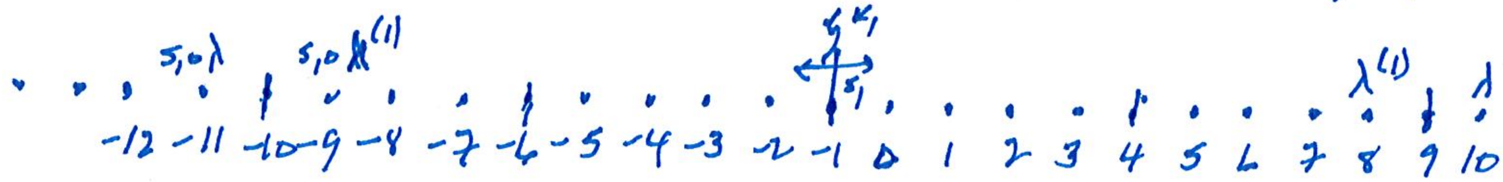
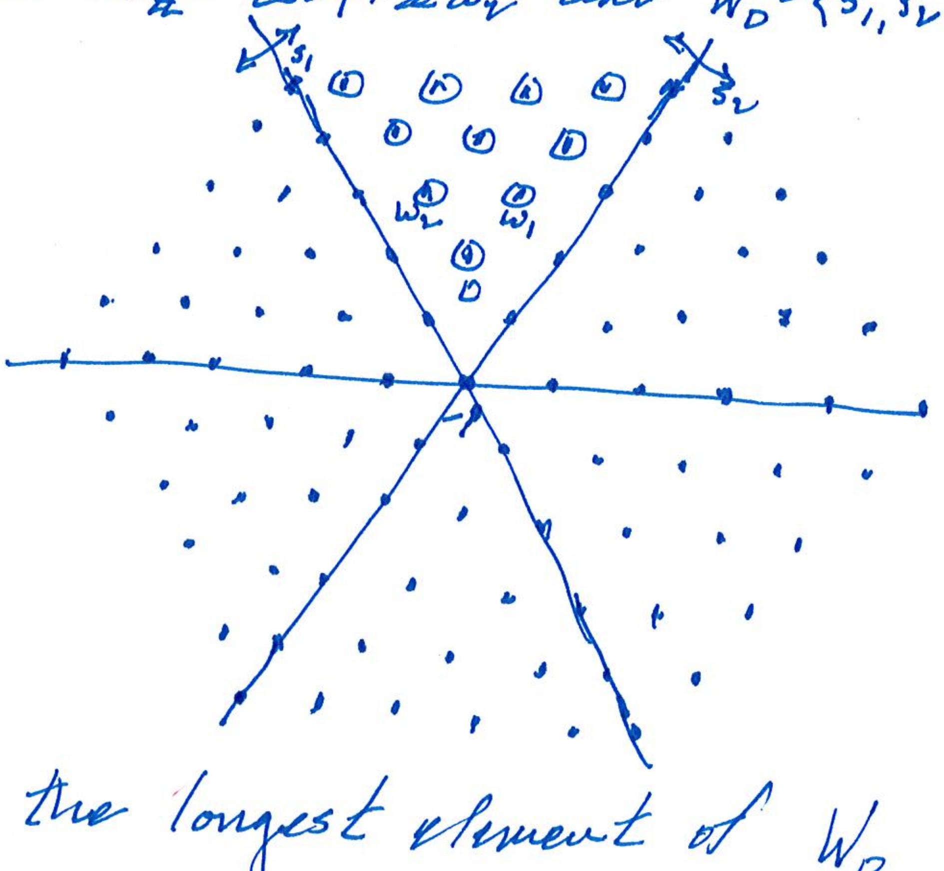
<u>Type SL_2</u> $\mathfrak{a}^*_{\mathbb{Z}} = \mathbb{Z}$ and $W_0 = \{1, s_1\}$ with $s_1^2 = 1$



$\langle \lambda+\rho, \alpha_i^\vee \rangle$ = distance from λ to $\mathfrak{h}^{\alpha_i}$.

$\lambda^{(1)} = \lambda - j\alpha_i$ if $\langle \lambda+\rho, \alpha_i^\vee \rangle = k\ell + j$ with $k \in \mathbb{Z}_{\geq 0}$, $\ell \in \{0, \ldots, \ell-1\}$

<u>Type SL_3</u> $\mathfrak{a}^*_{\mathbb{Z}} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $W_0 = \left\langle s_1, s_2 \;\middle|\; \begin{matrix} s_i^2 = 1 \\ s_1 s_2 s_1 = s_2 s_1 s_2 \end{matrix} \right\rangle$.



w_0 is the longest element of W_0

w_λ is the longest element of $W_\lambda = \mathrm{Stab}_{W_0}(\lambda)$

$(\mathfrak{a}^*_{\mathbb{Z}})^+ = \{ \lambda \in \mathfrak{a}^*_{\mathbb{Z}} \mid \langle \lambda+\rho, \alpha_i^\vee \rangle > 0 \} = \{ \odot \}$

Misra-Miwa Fock space KMS arXiv: q-alg 9508006

Type GL_n: $\mathfrak{a}^*_{\mathbb{Z}} = \mathbb{Z}^n$ and $W_0 = S_n$ = symmetric group.

Letting $V = \mathbb{C}^{\mathbb{Z}} = \mathbb{C}\text{-span}\{v_a \mid a \in \mathbb{Z}\}$ and

$$\Lambda^{\frac{\infty}{2}} V = \mathbb{C}\text{-span}\left\{ v_{a_1} \wedge v_{a_2} \wedge \cdots \;\middle|\; \begin{array}{l} a_j \in \mathbb{Z} \text{ and, for all but} \\ \text{a fin. number of } j, \\ a_j = -j+1 \end{array} \right\}$$

with

$$v_b \wedge v_a = \begin{cases} -(v_a \wedge v_b), & \text{if } a-b \in \ell\mathbb{Z}_{\geq 0} \\ -q(v_a \wedge v_b), & \text{if } 0 < a-b < \ell \\ -q(v_{b+j} \wedge v_{a-j}) - (v_{a-j} \wedge v_{b+j}) - q(v_a \wedge v_b) & \text{if } a-b = k\ell + j \text{ with } k \in \mathbb{Z},\ j \in \{0, \ldots, \ell-1\} \end{cases}$$

for $a < b$.

Then

$$\begin{aligned} \mathcal{F}_\ell &\longrightarrow \Lambda^{\frac{\infty}{2}} V \\ |\lambda\rangle &\longmapsto v_{\lambda_1 - 1 + 1} \wedge v_{\lambda_2 - 2 + 1} \wedge \cdots \end{aligned}$$

The <u>affine Lie algebra</u> is

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

with $\mathbb{Z}$-grading $\hat{\mathfrak{g}} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ given by

$$\deg(\mathfrak{g}t^k) = k, \quad \deg(K) = 0, \quad \deg(d) = 0.$$

<u>Parabolic category $\mathcal{O}$</u> <u>~~for~~ deg</u> is the category of $\hat{\mathfrak{g}}$-modules M such that

(a) M is $\mathfrak{g}_0$-semisimple

(b) M is $\mathfrak{g}_{\geq 0}$-locally nilpotent.

(b)-means: Let $\mathfrak{g}_{\geq 0} = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathfrak{g}_i$. If $m \in M$ then

$$\dim(U\mathfrak{g}_{\geq 0} \cdot m) < \infty.$$

~~Let $\{L(\lambda)$~~ M has <u>level $-\ell-h$</u> means:

if $m \in M$ then $K \cdot m = (-\ell-h)m$.

(h is the "dual Coxeter number".)

Let $\{L_0(\lambda) \mid \lambda \in (\mathfrak{a}_{\mathbb{Z}}^*)^+\}$ be the simple $\mathfrak{g}_0$-modules.

$$\Delta_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(\lambda) = U\hat{\mathfrak{g}} \otimes_{U\mathfrak{g}_{\geq 0}} L_0(\lambda). \qquad \text{"}\underline{\text{Verma module}}\text{"}$$

where $\mathfrak{g}_i \cdot L_0(\lambda) = 0$ if $i \in \mathbb{Z}_{>0}$.

$L(\lambda) = \Delta_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(\lambda) / \text{max proper}$.

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The affine Hecke algebra

The affine Weyl group W is generated by

$s_0, s_1, \ldots, s_n$ with $s_i^2 = 1$ and $\underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}}$

The affine Hecke algebra H is generated by

$T_i^2 = (t^{\frac12} - t^{-\frac12}) T_i + 1$

$T_0, T_1, \ldots, T_n$ with $T_i^2 = 1$ and $\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}}$

Define ε_0 a symbol with

$\varepsilon_0 T_i = (-t^{-\frac12}) T_i$ for $i = 1, 2, \ldots, n$

Define

$A_{-\varepsilon-\rho} = \{\triangle\}$ and $(\mathfrak{a}_{\mathbb{Z}}^*)^+ = \{\odot\}$

Define 1_v a symbol with

$T_j 1_v = t^{\frac12} 1_v$ if $s_j \in Stab_W(v)$.

H has bases

$\{T_w \mid w \in W\}$ and $\{X^w \mid w \in W\}$

with

$T_w = T_{i_1} \cdots T_{i_\ell}$ and $X^w = T_{i_1}^{\varepsilon_1} \cdots T_{i_\ell}^{\varepsilon_\ell}$

if $w = s_{i_1} \cdots s_{i_\ell}$ is min length

$$\varepsilon_j = \begin{cases} +1, & \text{if } j\text{th step of } s_{i_1}s_{i_2}\cdots \text{ is towards the sun} \\ -1, & \text{otherwise} \end{cases}$$

Define

$$[X_\lambda] = [X_{w_0 v_0 v}] = \varepsilon_0 X^{v} \mathbb{1}_v$$

where $w_0 v A_{L-h}$ is closest to the sun so that

$$\lambda = w_0 v_0 v$$

The sun is in the <u>antidominant chamber</u>

Define $\bar{\ }: H \to H$ by

$$\overline{t^{\frac{1}{2}}} = t^{-\frac{1}{2}} \quad \text{and} \quad \overline{T_i} = T_i^{-1}$$

and set $\overline{\varepsilon_0} = \varepsilon_0$ and $\overline{\mathbb{1}_v} = \mathbb{1}_v$.

Then $[C_\lambda]$ is determined by (for $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^+$)

$$\overline{[C_\lambda]} = [C_\lambda] \quad \text{and} \quad [C_\lambda] = [X_\lambda] + \sum_\mu p_{\lambda\mu} [X_\mu]$$

with $p_{\lambda\mu} \in t^{\frac{1}{2}}\mathbb{Z}[t^{\frac{1}{2}}]$.