"Geometric Peterson Isomorphism" Ottawa talk 30.04.2016 ①

Workshop on Equivariant generalized Schubert Calculus and applications | Arun Ram

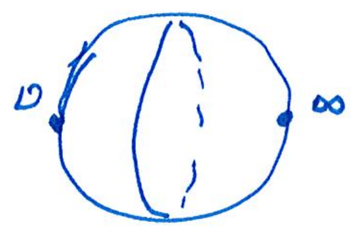
Curves in Flag Varieties

$$B = \left\{ \begin{pmatrix} a_1 & & b_{ij} \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \,\middle|\, \begin{matrix} a_i \in \mathbb{C}^\times \\ b_{ij} \in \mathbb{C} \end{matrix} \right\} \qquad B^- = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ c_{ij} & & a_n \end{pmatrix} \,\middle|\, \begin{matrix} a_i \in \mathbb{C}^\times \\ c_{ij} \in \mathbb{C} \end{matrix} \right\}$$

The flag variety is $GL_n(\mathbb{C})/B$.

Let $W_0 = \{\text{permutations}\}$. Then

$$GL_n(\mathbb{C}) = \bigsqcup_{u \in W_0} BuB = \bigsqcup_{v \in W_0} B^- vB.$$

$\mathbb{P}^1 =$ 

$$\mathcal{M}_3 = \{ c : \mathbb{P}^1 \to GL_n(\mathbb{C})/B \} = \mathrm{Mor}(\mathbb{P}^1, G/B).$$

$$\mathcal{M}_{3,\tau} = \{ c : \mathbb{P}^1 \to GL_n(\mathbb{C})/B \mid c_*([\mathbb{P}^1]) = \tau \}$$

where $\tau : H_*(\mathbb{P}^1) \to H_*(G/B)$.

$$\mathcal{M}_{3,\tau}^{u,v} = \left\{ c : \mathbb{P}^1 \to GL_n(\mathbb{C})/B \,\middle|\, \begin{matrix} c(0) \in BuB,\ c(\infty) \in B^- vB \\ c_*([\mathbb{P}^1]) = \tau \end{matrix} \right\}$$

so that

$$\mathcal{M}_3 = \bigsqcup_{\tau \in H_2(G/B)} \ \bigsqcup_{u,v \in W_0} \mathcal{M}_{3,\tau}^{u,v}.$$

Affine flag varieties

$GL_n(\mathbb{C}[t,t^{-1}])/I^+$ pos. affine flag variety

$GL_n(\mathbb{C}[t,t^{-1}])/I^0$ semi-infinite flag variety

$GL_n(\mathbb{C}[t,t^{-1}])/I^-$ neg. affine flag variety.

$$I^+ = \left\{ \begin{pmatrix} a_1 & & b_{ij} \\ & \ddots & \\ c_{ij} & & a_n \end{pmatrix} \;\middle|\; \begin{array}{l} a_i \in \mathbb{C}[t],\ a_i(0) \in \mathbb{C}^\times \\ b_{ij} \in \mathbb{C}[t] \\ c_{ij} \in t\mathbb{C}[t] \end{array} \right\}$$

$$I^- = \left\{ \begin{pmatrix} a_1 & & b_{ij} \\ & \ddots & \\ c_{ij} & & a_n \end{pmatrix} \;\middle|\; \begin{array}{l} a_i \in \mathbb{C}[t^{-1}],\ a_i(\infty) \in \mathbb{C}^\times \\ b_{ij} \in t^{-1}\mathbb{C}[t^{-1}] \\ c_{ij} \in \mathbb{C}[t^{-1}] \end{array} \right\}$$

$$I^0 = \left\{ \begin{pmatrix} a_1 & & b_{ij} \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \;\middle|\; \begin{array}{l} b_{ij} \in \mathbb{C}[t,t^{-1}] \\ a_i \in \mathbb{C}[t]^\times \end{array} \right\}$$

Let

$$\mathfrak{a}_{\mathbb{Z}} = \left\{ \begin{pmatrix} t^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & t^{\lambda_n} \end{pmatrix} \;\middle|\; \lambda_i \in \mathbb{Z} \right\} \quad \text{and } W = \mathfrak{a}_{\mathbb{Z}} \cdot W_0 .$$

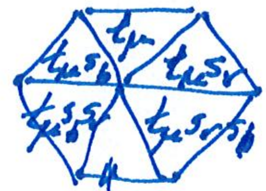
Then

$$GL_n(\mathbb{C}[t,t^{-1}]) = \bigsqcup_{x \in W} I^+ x I^+ = \bigsqcup_{y \in W} I^+ y I^- = \bigsqcup_{z \in W} I^+ z I^0 .$$

Alcove walks

tiles 

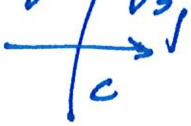
$W = \{\text{alcoves}\}$ $\qquad$ $\mathfrak{h}_{\mathbb{Z}} = \{\text{hexagons}\}$



$t_\mu s_b s_r s_b$.

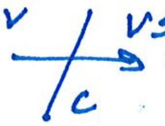
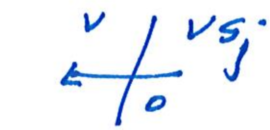
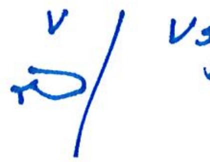
Let $j \in \{r, b, g\}$.

A <u>blackstep</u> of type j is



v is closer to 1 than vs_j.

A <u>purple step</u> of type j is

 or  or  .

$c \in \mathbb{C}$

v is closer to 1 than vs_j.

Let $x \in W$ and $x = s_{i_1} \cdots s_{i_\ell}$ a min length path to x.

$$I^+ x I^+ \longleftrightarrow \left\{ \begin{matrix} \text{black labeled walks of} \\ \text{type } i_1, \ldots, i_\ell \end{matrix} \right\}$$

$$I^+ x I^+ \cap I^- y I^+ \longleftrightarrow \left\{ \begin{matrix} \text{purple labeled walks of} \\ \text{type } i_1, \ldots, i_\ell \text{ that end} \\ \text{in } y \end{matrix} \right\}$$

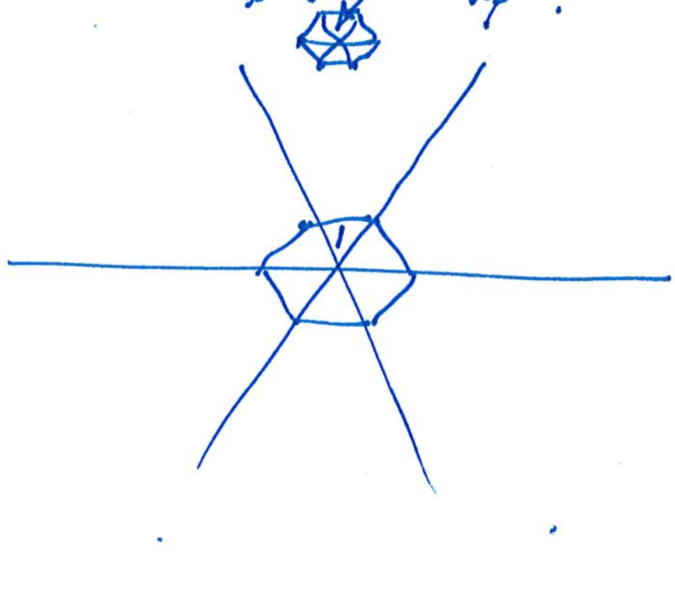
Quantum to affine

$$GL_n(\mathbb{C}[t,t^{-1}])/I^0 \xrightarrow{\sim} \mathcal{M}_3$$

More precisely, if $u,v\in W_0$ and $\tau\in\mathfrak{a}_{\mathbb{Z}}$ then

$$\begin{array}{ccc}
\mathcal{M}^{u,v}_{3,\tau} & & \begin{array}{l} c:\mathbb{P}^1 \to GL_n(\mathbb{C})/B \\ \quad c \mapsto g(c)B \end{array} \\
\updownarrow \wr & & \uparrow \\
I^+ u t_\mu I^0 \cap I^- v t_{\mu+\tau} I^0 & & g(t)I^0 = b^-(t^{-1}) v t_{\mu+\tau} I^0 \\
\updownarrow \wr & & \uparrow \\
I^+ u t_{\infty\rho^\vee} I^+ \cap I^- v t_{\infty\rho^\vee+\tau} I^- & & b^-(t^{-1}) v t_{\infty\rho^\vee+\tau} I^+
\end{array}$$

where $t_{\infty\rho^\vee}$ is a point deep in the dominant chamber $t_{\infty\rho^\vee}$.



<u>Theorem</u>

Let $u, v \in W_0$ and $\tau \in \mathfrak{a}_{\mathbb{Z}}$. Let

$$x = ut_{\omega\rho^\vee} \quad \text{and} \quad x = s_{i_1} \cdots s_{i_\ell}$$

a reduced word for x. Then

$$\mathcal{M}_3^{u,v} \longleftrightarrow \left\{ \begin{array}{l} \text{purple walks of type } i_1, \ldots, i_\ell \\ \text{that end at } v t_{\omega\rho^\vee + \tau}. \end{array} \right\}$$